SPECTRUM BROADCAST STRUCTURES FOR DISCRETE AND CONTINUOUS VARIABLES IN OPEN QUANTUM SYSTEMS

by

Alberto Acevedo

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As members of the Dissertation Committee, we certify that we have read the dissertation prepared by: Alberto Acevedo
titled:

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Jan Wehr
Jan Wehr (Nov 8, 2023 19:59 MST)
Jan Wehr — Date: Nov 8, 2023

Jaroslaw Korbicz
Jaroslaw Korbicz (Nov 8, 2023 22:49 GMT+1)
Jaroslaw Korbicz — Date: Nov 8, 2023

Moysey Brio
Moysey Brio (Nov 9, 2023 10:38 PST)
Moysey Brio — Date: Nov 9, 2023

Charles A. Stafford (Nov 13, 2023 19:18 MST)
Charles A. Stafford — Date: Nov 13, 2023

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Jan Wehr
Jan Wehr (Nov 8, 2023 19:59 MST)
Jan Wehr — Date: Nov 8, 2023
Jan Wehr

Jaroslaw Korbicz
Jaroslaw Korbicz (Nov 8, 2023 22:49 GMT+1)
Jaroslaw Korbicz — Date: Nov 8, 2023
Center For Theoretical Physics, Polish Academy Of Sciences

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Dedication

To my family...

To my brother David, for putting me on the path.

To my brother Danny, for reminding me to get off the path once in a while.

To my wife Cherie, for walking the path with me.

And most importantly...

To my mother, for making the path possible.

Contents

Abstract

We develop tools apt for the quantitative study of the dynamical emergence of Spectral Broadcast Structures (SBS) due to environmental monitoring. In the past, efforts have been made to bound the proximity of an arbitrary state undergoing non-unitary evolution to the nearest SBS state (in the trace distance sense). This dissertation presents the first of such bounds which has been substantiated as well as provides sufficient conditions under which a broad family of multipartite states converge to SBS states asymptotically in time. We also develop an SBS theory for continuous variables (SBSCV); i.e. the dynamics will now be generated by self-adjoint operators with purely continuous spectrum. We create a theory for SBSCV that parallels that of SBS and develop tools for its quantitative study.

Too often the mathematicians think it's physics, and the physicists think it's mathematics.

-Barry Simon.

Introduction

The emergence of the classical world, with its objective properties, from the quantum world has been a conundrum since the genesis of quantum theory [31][18][12]. If quantum mechanics is indeed a more fundamental theory of nature than classical mechanics (Newtonian mechanics), then why can we not see matter behaving *quantumly* in our everyday lives? The superposition principle says that states of matter may in a sense be delocalized [6] [69]. But we never see such delocalized states of matter in our quotidian life. So what happens to the *quantumness* of matter? Where does it all go? More importantly, why is the world around us classically objective (i.e. when multiple observers measure the same system they observe the same state of said system) in spite of it having to obey the laws of quantum mechanics which would appear to be highly non-objective? In this dissertation we study a relatively novel theory, known as the theory of Spectrum Broadcast Structures (SBS) theory [38] [39] [40] [53], developed for the purpose of taking on such questions. The theory of SBS being rather young (about 10 years old), it has yet to reach mathematical maturity. The bulk of the original work presented in this dissertation will consist of tools/techniques for the quantitative study of these so-called SBS states and their dynamical properties. We will also generalize the existing SBS theory to a new theory that includes states and dynamics hitherto not studied with mathematical rigor in the literature, namely the theory of SBS for continuous variables.

This dissertation contains four main parts. The first consists of a detailed synopsis of key concepts and tools from the theories of Quantum Open Systems, Quantum Information Theory and Operator Theory that will be essential for the development of the main results of this work. Chapters 1 and 2 will be consecrated to the introduction of all of these concepts and tools. In Chapter 1 we introduce the concepts of quantum open systems, quantum information theory, and operator Theory; focusing on Quantum Maps, distance measures, and relevant operator inequalities in Chapter 2.

The second part (Chapter 3) introduces the optimization problem of Quantum State Discrimination (QSD) [22][23][70][61][25][27][65], an active field of research within the theory of quantum information whose focus is estimating the probability of measuring the correct state of a quantum system when said system is in a mixed state. This section will serve us in future chapters since techniques and concepts from QSD will play an important role in Chapters 4 and 5. In Chapter 3 we introduce some of the main results of QSD for countable mixtures and introduce the concept of QSD for uncountable mixtures. We study dynamically evolving mixtures, both uncountable and countable, and deduce novel results regarding their QSD problems in the asymptotic regime (large-time dynamics); see Propositions 3.5.1 and 3.7.1, Theorem 3.6.2, and Corollary3.6.1.

In Chapter 4 (Part 3 of this thesis) we focus on the theory of Spectrum Broadcast Structures (SBS) [38] [39] [40] [53]. Informally, a SBS state is a special type of quantum state that exhibits classical-objectivity properties. As mentioned, these states are used to study the emergence of classicality from the quantum, a theme that is ubiquitous in the field of *quantum-to-classical transitions* [12]. Interacting systems such as a multipartite network of a system and multiple observers (observers of the system) will always exhibit objectivity in the classical regime; i.e. all observers will find the system to be in the same state. One would expect an analog of such a notion of objectivity to exist in the quantum regime. In Chapter 4 it will be argued that in order to prove the latter is indeed the case, quantum analogs of the networks hitherto discussed (i.e. one system with multiple observers.) will necessarily have to converge dynamically to an SBS state (4.1.2. We build tools apt for studying the dynamical convergence of certain multipartite quantum states to an SBS state (Theorems 4.2.2 and (4.4.1)) and discuss sufficient conditions for a broad family of multi-partite states to converge to an SBS state (see Corollary 4.8.1)

Chapter 5 constitutes the fourth and final part of this dissertation. In this chapter, we generalize the theory of SBS to include the case where the systems in question are taken to live in an infinite dimensional Hilbert space, and the dynamics are assumed to be generated by self-adjoint operators with continuous spectrum (what we will call the Spectrum Broad Cast Structures of continuous variables theory (SBSCV)). Such cases are not supported by the already existent theory of SBS which shall be discussed in Chapter 4 and the novel results presented therein. The leap in difficulty from SBS to SBSCV is considerable; we, therefore, consecrate a lot of the space in Chapter 5 to the formidable task of creating the tools necessary for the proper analysis of the emergence of SBSCV states (see 5.4.1 5.3.1 5.3.1).

Chapter 1

Open Quantum Systems

1.1 Closed vs Open

The central focus of introductory quantum mechanics is the study of closed quantum systems. Although the closedness of any physical system is an idealization, and in reality no physical is ever closed except perhaps for the entire universe [1], assuming closedness may nevertheless lead to useful models that shed light on fundamental properties of nature. The Energy spectrum of the hydrogen atom, for example, may be deduced by considering the hydrogen to be a closed system [6]. The equation of motion governing closed quantum systems is the Schrödinger equation $i\hbar\partial_t|\psi_t\rangle = \hat{\mathbf{H}}|\psi_t\rangle$, to be discussed in detail below. Schr$\ddot{o}$dinger's equation (SE) generates unitary evolution, forgoing any description of dissipative and decoherence effects [12]; processes where energy, information, and other agents of "quantumness" are not preserved. Decoherence and dissipation are nevertheless inevitable! A model that does not account for such a phenomenon can therefore not be complete.

One of the conundrums out-flowing from the modeling of quantum scale phenomena with the SE is the seeming non-physical nature of its solutions, in the sense that linear combinations of measurable properties of an observable are viable solutions to SE. This is of course the superposition principle of quantum mechanics [6], something whose interpretation foments contention amongst scientists till this day. The superposition principle [6] [10] is arguably the needy-greedy when one discusses quantum mechanical phenomenon, and indeed this is what is alluded to when the term "quantumness" is used in this work. In everyday macroscale physical experiences, one does not carry out measurements that result in more than one outcome, hence the dilemma.

The non-local description of nature emanating from early quantum theory was not met with a favorable reception. In fact, quite the opposite, the immediate impulse of some of the great physicists of the 20th century was to navigate around this apparent inconstancy of quantum theory and our apparent reality. Consensus was more or less met but there was one question that still left the physics community puzzled. This was the question of measurement, i.e. if quantum theory is correct then how is it the case that measurements always lead to definite outcomes? To answer this question Niels Bohr annexed a supporting theory to the already-existent quantum theory. The idea consisted of a model where upon being measured, a quantum system evolving unitarily via SE would instantaneously collapse to a definite eigenstate of the observable being measured. This collapse was assumed to occur with some probability distribution governing the "collapse"[9]. The latter take on quantum theory became known as the Copenhagen interpretation [7]. Although various aspects of the Copenhagen interpretation are criticizable, we will stop at one, namely the instantaneous aspect of the collapse which is assumed. This is at odds with the nature of the time-continuous world we live in, there must surely exist some time scale within which these "collapses" take place, leading one to guess that the Copenhagen school of thought has missed something fundamental. It is now understood that "collapse" or loss of quantumness, is not a phenomenon that occurs instantaneously. This loss of quantumness is the study of the theory of decoherence [12] [18] and decoherence time scales which vary from system to system [18] (pg 66) may be estimated via considering the more realistic setting where the system in question, and the associated environment, are both treated as interacting quantum systems.

By considering a larger closed quantum system, which includes both the system of interest and its environment, one may use SE as a starting point, and later compute the reduced dynamics of the system. Ideally, we hope to obtain the unitary evolution for the total dynamics and deduce the local dynamics of the system. However, as will be seen in what is to come, this is no easy venture. In this chapter, we will give an overview of the mathematical theory for closed quantum systems, introduce the theory of open quantum systems, and discuss applications to canonical decoherence models of decoherence [18] [12].

1.2 Unitary Evolution For Finite Dimensional Systems

In quantum mechanics for closed quantum systems one is tasked with solving SE; we will express it here with units of $\hbar = 1$.

$$i\partial_t|\psi_t\rangle = \hat{\mathbf{H}}|\psi_t\rangle \quad (\hbar = 1) \tag{1.1}$$

where the state $|\psi_t\rangle$ is a vector in some *Hilbert* space, call it $\hat{\mathscr{H}_S}$, and $\hat{\mathbf{H}}$ is a self-adjoint operator, which we will call the *Hamiltonian*, acting in $\mathscr{H}_S$. For the cases where $\mathbf{dim}\{\mathscr{H}_S\} < \infty$, $|\psi_t\rangle$ will represent a column vector, making $\langle\psi_t| := |\psi_t\rangle^\dagger$ a row vector. The solution to equation 1.1 is

$$e^{-it\hat{\mathbf{H}}}|\psi_0\rangle \tag{1.2}$$

where $|\psi_0\rangle \in \mathscr{H}_S$ ($\langle\psi_0|\psi_0\rangle = 1$) is the initial state of the system. Invoking Stone's theorem [3] we know that the operator $e^{-it\hat{\mathbf{H}}}$ is a strongly continuous one-parameter semigroup acting in the Hilbert space $\mathscr{H}_S$, with t being the parameter, and hence $\langle\psi_t|\psi_t\rangle = 1$ $(\forall t)$. However, the story does not end there. We would like to understand the effect of the unitary operator $\hat{\mathbf{U}}_t := e^{-it\hat{\mathbf{H}}}$ upon acting on $|\psi_0\rangle$. One approach is to simply use the Taylor expansion of the exponential function e^x.

$$e^{-it\hat{\mathbf{H}}} = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}\hat{\mathbf{H}}^n \tag{1.3}$$

However, this will require one to take arbitrarily large powers of the matrix $\hat{\mathbf{H}}$. Inevitably, there will be some approximations involved since one can not take an infinite sum, and taking greater powers of $\hat{\mathbf{H}}$ becomes ever more costly. There are nevertheless cases where this path is fruitful; one of them being when the operator norm $\|t\hat{\mathbf{H}}\|$ is small. In such a case one can get away with truncating the sum and keeping a minimal amount of terms. Or, the Hamiltonian could have small dimensions, in which case computing powers are not too costly.

There is one case where the sum (1.3) is trivial to compute. This is when the matrix $\hat{\mathbf{H}}$ is diagonal. Assume that $\hat{\mathbf{H}}$ is diagonal with respect to the orthonormal basis $\{|\phi_n\rangle\}_n$, then

$$\hat{\mathbf{H}} = \sum_n \lambda_n |\phi_n\rangle\langle\phi_n| \tag{1.4}$$

and therefore

$$e^{-it\hat{\mathbf{H}}} = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \hat{\mathbf{H}}^n = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \left(\sum_m \lambda_m |\phi_m\rangle\langle\phi_m| \right)^n = \tag{1.5}$$

$$\sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \sum_m \lambda_m^n |\phi_m\rangle\langle\phi_m| = \sum_m \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \lambda_m^n |\phi_m\rangle\langle\phi_m| = \tag{1.6}$$

$$\sum_{n=0}^{\infty} \frac{(-it)^n}{n!} \sum_m \lambda_m^n |\phi_m\rangle\langle\phi_m| = \sum_m e^{-it\lambda_m} |\phi_m\rangle\langle\phi_m|. \tag{1.7}$$

Assuming that the operator $\hat{\mathbf{H}}$ is full rank, its eigenspace $\{|\phi_n\rangle\}_n$ will span all of $\mathscr{H}_S$; hence we need only decompose any arbitrary vector $|\psi_0\rangle$ in $\mathscr{H}_S$ with respect to the eigenbasis of $\hat{\mathbf{H}}$ and then calculate exactly the evolution of $|\psi_0\rangle$. i.e.

$$\hat{\mathbf{U}}_t |\psi_0\rangle = e^{it\hat{\mathbf{H}}} \sum_n \alpha_n |\phi_n\rangle = \sum_n \alpha_n e^{it\lambda_n} |\phi_n\rangle, \quad \{\alpha_k := \langle\phi_k|\psi_0\rangle\}_k \tag{1.8}$$

It is a basic result from linear algebra that any *normal* matrix $\hat{\mathbf{H}}$ ($\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}} = \hat{\mathbf{H}}\hat{\mathbf{H}}^\dagger$) is diagonalizable [9] (section 2.1.7). Owing to the fact that self-adjoint matrices are also normal, we conclude that any self-adjoint matrix $\hat{\mathbf{H}}$ is also diagonalizable. Obtaining a diagonal representation, however, remains a challenging problem. In this case we must solve the following vector equation.

$$\hat{\mathbf{H}}|\phi\rangle = \lambda|\phi\rangle \tag{1.9}$$

which is only possible if the matrix $\hat{\mathbf{H}} - \lambda\mathbb{I}$ is singular. From basic linear algebra, we know that ensuring the matrix $\hat{\mathbf{H}} - \lambda\mathbb{I}$ to be singular is equivalent to finding the roots of the characteristic polynomial

$$det\left(\hat{\mathbf{H}} - \lambda\mathbb{I}\right) = 0. \tag{1.10}$$

The hurdles are now that of computing a determinant, finding the roots of the resulting polynomial in order to obtain the eigenvalues, and finding the associated eigenvectors. These three processes are well understood theoretically; computationally they are costly and even intractable for large enough $\mathbf{dim}\{\mathscr{H}_S\}$. Although the equation (1.10) in general poses an ill-conditioned problem there are many algorithmic techniques designed for the computational estimation of the solutions to this equation [54].

We have of course omitted discussions on the slew of eigenvalue approximation techniques that exist out there. Perhaps the most popular one amongst physicists being time-dependent perturbation theory. Nevertheless, the discussion herein, in a sense, summarizes the mathematical aspects of finite dimensional quantum theory for closed quantum systems. At a mathematical level, if we can diagonalize the Hamiltonian of the system we are interested in, then we know just about everything. Of course, anyone who has read more than the introductory chapters to a first-year quantum mechanics course knows that even the simplest non-trivial *Hilbert* space $\mathbb{C}^2$, simple as it may be, affords a framework that supports volumes of interesting physics[10] [6] [13] [9] [8]. Two-level systems are the basis of quantum information and quantum computation theory after all[9], and two-level systems are just the tip of the iceberg!

1.3 Unitary Evolution For Infinite Dimensional Systems

Let us now pass to the case where $\mathbf{dim}\{\mathscr{H}_S\} = \infty$. Unless mentioned otherwise, we will constrain ourselves to the Hilbert space $L^2(\mathbb{R})$ (square-integrable functions over the reals) for this section. In quantum theory, observables are described with self-adjoint operators. The expectation value of some arbitrary observable $\hat{\mathbf{H}}$ for some system in the state $|\psi\rangle$ being $\langle\hat{\mathbf{H}}\rangle_\psi := \langle\psi|\hat{\mathbf{H}}|\psi\rangle := \langle\psi, \hat{\mathbf{H}}\psi\rangle$; $|\psi\rangle$ now represent square-integral functions $\psi(x)$ and $\langle\psi|$ their complex conjugate $\psi^*(x)$. Physically speaking, if the state of the system we are studying lives in $L^2(\mathbb{R})$, we are only going to concern ourselves with states $|\psi\rangle$ that yield defined expectation values for all observables of the system. For the case of some observable $\hat{\mathbf{H}}$ acting $L^2(\mathbb{R})$, we hence know that

$$\langle\hat{\mathbf{H}}\rangle_\psi := \int_{\mathbb{R}} \psi^*(x)\hat{\mathbf{H}}\psi(x)dx \tag{1.11}$$

is a well-defined quantity. If the operator $\hat{\mathbf{H}}$ is bounded then we have nothing to worry about since $\langle\hat{\mathbf{H}}\rangle_\psi \leq \|\hat{\mathbf{H}}\| < \infty$, which follows from the result $sup_{\|x\|=1}\|\langle x, \hat{\mathbf{A}}x\rangle| = \|\hat{\mathbf{A}}\|$, for a bounded and self-adjoint $\hat{\mathbf{A}}$, consult [21] chapter 8. For the case where the observable in question, $\hat{\mathbf{H}}$, is an unbounded operator the latter will not be the case for all $|\psi\rangle$. However, the well-deifinedness (1.11) can be guaranteed by restricting ourselves to $|\psi\rangle$ satisfying $\hat{\mathbf{H}}|\psi\rangle \in L^2(\mathbb{R})$, i.e. $\int_{\mathbb{R}} |\hat{\mathbf{H}}\psi(x)|^2 dx < \infty$. This is immediately clear by making use of the Cauchy-Schwarz inequality;

$$\int_{\mathbb{R}} \psi^*(x)\hat{\mathbf{H}}\psi(x)dx \leq \sqrt{\int_{\mathbb{R}} |\psi^*(x)|^2 dx}\sqrt{\int_{\mathbb{R}} |\hat{\mathbf{H}}\psi(x)|^2 dx} = \sqrt{\int_{\mathbb{R}} |\hat{\mathbf{H}}\psi(x)|^2 dx}. \tag{1.12}$$

Let us call the subset of $|\psi\rangle \in L^2(\mathbb{R})$ satisfying $\hat{\mathbf{H}}|\psi\rangle \in L^2(\mathbb{R})$ the domain of $\hat{\mathbf{H}}$, and name it $\mathcal{D}(\hat{\mathbf{H}}) \subset L^2(\mathbb{R})$. In quantum theory, we are only concerned with Hamiltonians $\hat{\mathbf{H}}$ having a domain $\mathcal{D}(\hat{\mathbf{H}})$ which is dense in the Hilbert space of interest; i.e. any state in the Hilbert space of interest may be approximated by elements of $\mathcal{D}(\hat{\mathbf{H}})$. This is a point in which the mathematics for infinite-dimensional quantum systems diverges from that of finite-dimensional systems. For physical systems living in a finite-dimensional Hilbert space, one need not worry about domain issues; the inner product (1.12) in such a case will simply be a finite sum of finite values and will therefore be well-defined regardless of the operator and vectors involved. Furthermore the domain of some observable $\hat{\mathbf{A}}$ in the finite-dimensional case may always be treated as the entire Hilbert of interest.

Any operator $\hat{\mathbf{H}}$ acting in $L^2(\mathbb{R})$ will have a domain $\mathcal{D}(\hat{\mathbf{H}})$ and its adjoint $\hat{\mathbf{H}}^\dagger$ will have its own corresponding domain $\mathcal{D}(\hat{\mathbf{H}}^\dagger)$, $\mathcal{D}(\hat{\mathbf{H}}^\dagger) \subset L^2(\mathbb{R})$ and $\mathcal{D}(\hat{\mathbf{H}}) \subset L^2(\mathbb{R})$. For an operator $\hat{\mathbf{H}}$ in such a setting to be self-adjoint it is a necessary and sufficient condition that $\mathcal{D}(\hat{\mathbf{H}}) = \mathcal{D}(\hat{\mathbf{H}}^\dagger)$. Drawing a comparison with the finite-dimensional case, notice that for the latter one need only check that the matrix in question is equal to its complex transpose. Even at the level of identifying whether or not an operator $\hat{\mathbf{H}}$, which we would like to use for the modeling of some observable, is self-adjoint, one is met with a challenge far greater than just simple matrix manipulation. In the non-relativistic quantum theory of closed quantum systems, the operators of interest have the following structure

$$\hat{\mathbf{H}} = \hat{\mathbf{H}}_0 + V(\hat{\mathbf{X}}) \tag{1.13}$$

where $\hat{\mathbf{H}}_0 := -\sum_k \frac{1}{2m_k}\partial^2_{x_k}$ and $\hat{\mathbf{X}} := (\hat{\mathbf{x}}_1, ..., \hat{\mathbf{x}}_k, ..)$. Testing whether or not $\mathcal{D}(\hat{\mathbf{H}}) = \mathcal{D}(\hat{\mathbf{H}}^\dagger)$ is of course not a simple matter, but thanks to theorems such as the Kato-Rellich [4] the self-adjointness of operators of the form (1.13) is fairly understood. When an operator of the form (1.13) is self-adjoint, it is referred to as a *Schrödinger operator* [5].

Let us now look at a concrete example. We will consider what is perhaps the most famous observables in quantum theory, namely $\hat{\mathbf{P}} := -i\partial_x$ (the momentum operator). We remind the reader that the Hilbert space of interest shall be $L^2(\mathbb{R})$. In such a case it is easy to see that that there exists $|\psi\rangle \in L^2(\mathbb{R})$ such that $-i\partial_x\psi(x) \notin L^2(\mathbb{R})$. For example, let $\psi(x) = \frac{\sin(e^x)}{1+x^2}$. This function is square-integrable since it is dominated by an $L^2(\mathbb{R})$ function $\frac{1}{1+x^2}$, i.e. $|\psi(x)| \le \frac{1}{1+x^2}$ $\forall x \in \mathbb{R}$ (the argument follows from application of the Dominated Convergence Theorem [2]). However, consider the derivative of $\psi(x)$.

$$\partial_x\psi(x) = \frac{(1+x^2)\cos(e^x)e^x - \sin(e^x)2x}{(1+x^2)^2}. \tag{1.14}$$

For large x, $|\partial_x\psi(x)| \approx \frac{e^x}{1+x^2}$; a function which is clearly not in square-integrable. However, $\mathcal{D}(\hat{\mathbf{P}})$ is a dense subset of $L^2(\mathbb{R})$, which means that any physically relevant function living in $L^2(\mathbb{R})$ may be approximated by functions in the domain of $\hat{\mathbf{P}}$. This will allows us to make sense of $\langle\hat{\mathbf{P}}\rangle_\psi$ for any physically relevant $|\psi\rangle$.

Given a Hamiltonian $\hat{\mathbf{H}}$ acting in an infinite dimensional Hilbert space, Schrödinger's equation (1.1) again has the solution $e^{-it\hat{\mathbf{H}}}|\psi_0\rangle$, with $|\psi_0\rangle$ the initial state. This time however, the operator $\hat{\mathbf{H}}$ has domain restrictions which may be transmitted to the operator $e^{-it\hat{\mathbf{H}}}$. Deffering nuances regarding domains to the great expositions in [4] [3], the generalization of Stone's theorem tells us that the operator $e^{-it\hat{\mathbf{H}}}$ will again be a unitary operator governing the dynamics of any initial state $|\psi_0\rangle$. Here we are once again faced with the challenge of figuring out how the operator $e^{-it\hat{\mathbf{H}}}$ acts on an arbitrary state $|\psi_0\rangle$. Once again, one might be tempted to expand the operator $e^{-it\hat{\mathbf{H}}}$ using the

Taylor series of e^x. If we were to do this, even for a simple case of the Schrödinger operator (1.13), say $\hat{\mathbf{H}} = -\frac{1}{2m}\partial_x^2 + V\big(\hat{\mathbf{X}}\big)$, we would quickly encounter non-linear terms such as $\partial_x^2 V\big(\hat{\mathbf{X}}\big)\psi_0(x)$. These non-linear terms would grow in complexity and we would have to compute an indefinite amount of them. Also, given that the operations involved are derivatives and multiplication by functions, we would not be able to simply take various powers of the operators showing up in the series, e.g. $\big\{\partial_x^2\big\}^m\big\{V\big(\hat{\mathbf{X}}\big)\big\}^n$, and store them in some computer program subroutine for computing the effect on an arbitrary $\big|\psi_0\big\rangle$. This is because the way differential operators act will be intrinsically dependent on the vector they are acting on. We are worst off here than we were in the finite-dimensional case. Unless t is very small and we can truncate the series, one must take a different approach.

To make progress in the matter of making sense of $e^{-it\hat{\mathbf{H}}}\big|\psi_0\big\rangle$ for the case at hand, we need a notion that generalizes the concept of an eigenvalue and eigenvector. This is the notion of the *spectrum* of an operator denoted $\mathbf{Spec}\big\{\hat{\mathbf{H}}\big\}$ for some operator $\hat{\mathbf{H}}$[3] [4] [16] [8]. Given that we are solely interested in self-adjoint operators, we will be narrowing our attention to the spectral theory of self-adjoint operators. We warn the reader that our treatment will be a non-rigorous one. There are four main kinds of infinite-dimensional self-adjoint operators that we shall be working with in this thesis. Namely, those that are *trace class*, *Hilbert-Schmidt*, bounded and unbounded operators. Given an infinite-dimensional Hilbert space $\mathscr{H}$ we will denote the corresponding spaces of bounded, trace class, and Hilbert-Schmidt operators respectively as $\mathcal{B}\big(\mathscr{H}\big)$, $\mathscr{S}_1\big(\mathscr{H}\big)$ and $\mathscr{S}_2\big(\mathscr{H}\big)$.

$$\mathcal{B}\big(\mathscr{H}\big) := \Big\{\hat{\mathbf{H}} : \mathscr{H} \to \mathscr{H} \;\Big|\; \|\hat{\mathbf{H}}\| < \infty\Big\} \tag{1.15}$$

$$\mathscr{S}_2\big(\mathscr{H}\big) := \Big\{\hat{\mathbf{H}} : \mathscr{H} \to \mathscr{H} \;\Big|\; \sqrt{Tr\{\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}\}} < \infty\Big\} \tag{1.16}$$

$$\mathscr{S}_1\big(\mathscr{H}\big) := \Big\{\hat{\mathbf{H}} : \mathscr{H} \to \mathscr{H} \;\Big|\; Tr\big\{\sqrt{\hat{\mathbf{H}}^\dagger\hat{\mathbf{H}}}\big\} < \infty\Big\} \tag{1.17}$$

It is useful to note that

$$F\big(\mathscr{H}\big) \subset \mathscr{S}_1\big(\mathscr{H}\big) \subset \mathscr{S}_2\big(\mathscr{H}\big) \subset \mathcal{B}\big(\mathscr{H}\big) \tag{1.18}$$

where $F\big(\mathscr{H}\big)$ are the finite rank matrices acting in $\mathscr{H}$. Above, the map $Tr\{\}$ is the trace of an operator, which is equivalent to the sum of the eigenvalues of the operator being traced[16]. There are two things that the attentive reader might have noticed. The first one is that there is no mention of the case for unbounded operators in the list above. This is because, unlike the sets $\mathcal{B}\big(\mathscr{H}\big), \mathscr{S}_2\big(\mathscr{H}\big), \mathscr{S}_1\big(\mathscr{H}\big), F\big(\mathscr{H}\big)$ which are a special type of *Banach* space called a *Banach* algebra [3], unbounded operators do not form an algebra, nor a linear space, because each is defined in its own domain. The second thing one might have noticed is that there was no mention of compact operators. Indeed the space of trace class operators as well as the space of Hilbert-Schmidt operators are families of compact operators. However, we will seldom be needing the concept of a compact operator in any greater generality than that of trace class and Hilbert-Schmidt operators.

Heuristically speaking, finding the spectrum of some self-adjoint operator in the infinite-dimensional case is paramount to finding objects $|s\rangle$ so that the relationship

$$\hat{\mathbf{H}}|s\rangle = \mu_s|s\rangle \tag{1.19}$$

holds. The problem here is that in this case the objects $|s\rangle$ may not be elements of the Hilbert space in question. They may belong to the space of distributions for example. e.g. consider the position operator $\hat{\mathbf{X}}$. This is an unbounded operator which may satisfy an equation akin to (1.19). Namely,

$$\hat{\mathbf{X}}|x\rangle = x|x\rangle \tag{1.20}$$

where the objects $|x\rangle$ are Dirac delta distributions written in ket form, we will refer to such eigenstates as generalized eigenstates; one might correctly guess that $\mathbf{Spec}\{\hat{\mathbf{X}}\} = \mathbb{R}$ in this case. However, $\delta(x)$ is not in $L^2(\Omega)$ for any $\Omega \subseteq \mathbb{R}$, so it would seem that this relationship will not be too useful. However, taking the physicist approach, one may write $\hat{\mathbf{X}} = \int_{\mathbb{R}} x|x\rangle\langle x|dx$ (Spectral Theorem [3]), and noticing that $\hat{\mathbf{X}}^n = \int_{\mathbb{R}} x^n|x\rangle\langle x|dx$ one may therefore conclude that

$$e^{-it\hat{\mathbf{X}}} = \int_{\mathbb{R}} e^{-itx}|x\rangle\langle x|dx \tag{1.21}$$

Now, for an arbitrary $|\psi\rangle \in L^2(\mathbb{R})$ we may operate with (1.21) to get

$$e^{-it\hat{\mathbf{X}}}|\psi\rangle = \int_{\mathbb{R}} e^{-itx}|x\rangle\langle x|\psi\rangle dx = \int_{\mathbb{R}} \psi(x)e^{-itx}|x\rangle dx \tag{1.22}$$

which is just the $L^2(\mathbb{R})$ function $e^{-itx}\psi(x)$. But this is what we expected since $e^{-it\hat{\mathbf{X}}}$ was evidently a multiplication operator to begin with, and we can always write our sates as functions of $x \in \mathbf{Spec}\{\hat{\mathbf{X}}\}$. Another example from physics is the quantum simple harmonic oscillator (QSHO). Such a system is described by the Schrödinger operator

$$\hat{\mathbf{H}} := \frac{1}{2m}\hat{\mathbf{P}}^2 + \frac{m\omega^2}{2}\hat{\mathbf{X}}^2. \tag{1.23}$$

Although quite innocuous-looking, such an operator does not lend itself to exponentiation right away. One needs to find its spectrum and associated spectral decomposition akin to what we saw in (1.21). To do this we must solve an equation of the form

$$\left(\frac{1}{2m}\hat{\mathbf{P}}^2 + \frac{m\omega^2}{2}\hat{\mathbf{X}}^2\right)|s\rangle = \mu_s|s\rangle. \tag{1.24}$$

Forgetting for a moment that the solutions for (1.24) are amongst the most widely known in quantum theory, one must admit that solving this is no more straightforward than solving a differential equation. We were lucky in the case of $\hat{\mathbf{X}}$, where the spectrum was trivial to find. For the general case, some clever methods must be devised; such is the case for the QSHO. Via clever manipulation of ladder operators, it was discovered that $\hat{\mathbf{H}}$ could be diagonalized with *Hermite* functions, also known as number states, $|n\rangle$ with respective eigenvalues $\omega(n+\frac{1}{2})$ [6] [45] [10]. Where

$$\langle x|n\rangle := \frac{1}{\sqrt{2^n\, n!}}\left(\frac{m\omega}{\pi}\right)^{1/4} e^{-\frac{m\omega x^2}{2}} H_n\left(\sqrt{m\omega}x\right) \ (\textit{Hermite Functions}) \tag{1.25}$$

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right) \ (\textit{Hermite Polynomials}) \tag{1.26}$$

With this in mind, we can now write

$$e^{-it\left(\frac{1}{2m}\hat{\mathbf{P}}^2+\frac{m\omega^2}{2}\hat{\mathbf{X}}^2\right)} = \sum_{n=0}^{\infty} e^{-it\omega(n+\frac{1}{2})}|n\rangle\langle n|. \tag{1.27}$$

Let us compare (1.22) and (1.27). Both operators are diagonal in a sense, with respect to their eigenvectors and generalized eigenvectors. We know exactly how (1.27) and (1.21) act on any given vector in their respective domains. What differentiates these two operators is the type of spectrum they have. While $\mathbf{Spec}\{\hat{\mathbf{X}}\} = \mathbb{R}$, we have $\mathbf{Spec}\{\frac{1}{2m}\hat{\mathbf{P}}^2 + \frac{m\omega^2}{2}\hat{\mathbf{X}}^2\} = \{\omega(n+\frac{1}{2})\}_{n=0}^{\infty}$. The latter is a set of isolated points while the former, $\mathbb{R}$, is a connected set! These are cases of what is known as absolutely continuous and point spectrum respectively. The spectrum of any self-adjoint operator may be characterized by three subcategories.

Definition 1.3.1 (The Spectrum of an operator)

Let $\hat{\mathbf{H}}$ be an arbitrary operator acting over some *Hilbert* space $\mathscr{H}$; the spectrum of said operator is the union of following complimentary sets.

- **point spectrum** of $\hat{\mathbf{H}} := \overline{\mathbf{Spec}}_{pp}(\hat{\mathbf{H}})$: The closure of the set of eigenvalues of $\hat{\mathbf{H}}$, i.e. $\mathbf{Spec}_{pp}(\hat{\mathbf{H}})$ is the set of eigenvalues of $\hat{\mathbf{H}}$ (this is called the pure point spectrum [3]).
- **continuous spectrum** of $\hat{\mathbf{H}} := \mathbf{Spec}_c(\hat{\mathbf{H}})$: Consists of all scalars, λ that are not eigenvalues but make the range of $\hat{\mathbf{H}} - \lambda\mathbb{I}$ a proper dense subset of $\mathscr{H}$.
- **residual spectrum** of $\hat{\mathbf{H}} := \mathbf{Spec}_r(\hat{\mathbf{H}})$: $\hat{\mathbf{H}} - \lambda\mathbb{I}$ is injective but does not have dense range.

It is important to note that all of the sets defined in Definition 1.3.1 are disjoint, whence the spectrum of an arbitrary operator $\hat{\mathbf{H}}$ may expressed as $\mathbf{Spec}(\hat{\mathbf{H}}) = \overline{\mathbf{Spec}}_{pp}(\hat{\mathbf{H}}) \cup \mathbf{Spec}_c(\hat{\mathbf{H}}) \cup \mathbf{Spec}_r(\hat{\mathbf{H}})$. Note that for the case were $\hat{\mathbf{H}}$ is self-adjoint $\mathbf{Spec}_r(\hat{\mathbf{H}})$ is empty [3]. Hence, when for self-adjoint $\hat{\mathbf{H}}$, $\mathbf{Spec}(\hat{\mathbf{H}}) = \overline{\mathbf{Spec}}_{pp}(\hat{\mathbf{H}}) \cup \mathbf{Spec}_c(\hat{\mathbf{H}})$. There are indeed other ways to partition the spectrum of a self-adjoint operator but we will stop at this for now, recommending to the interested reader the discussion in chapter 9 of [4] for yet another partitioning of the spectrum using the notions of *discrete* and *essential* spectrums. In Section 5.8 we will introduce yet another decomposition of the spectrum that will involve two types of continuous spectrum, *absolutely continuous* and *singular continuous*.

Returning to the position operator $\hat{\mathbf{X}}$ and the QSHO $\frac{1}{2m}\hat{\mathbf{P}}^2 + \frac{m\omega^2}{2}\hat{\mathbf{X}}^2$, note that for the latter case the equation

$$\big(\frac{1}{2m}\hat{\mathbf{P}}^2 + \frac{m\omega^2}{2}\hat{\mathbf{X}}^2\big)\big|n\big\rangle = \omega(n+\frac{1}{2})\big|n\big\rangle \tag{1.28}$$

is a bonafide solution to the eigenvalue problem since the states $\big|n\big\rangle$ are square-integrable functions. Therefore it is clear that $\mathbf{Spec}_p\big\{\frac{1}{2m}\hat{\mathbf{P}}^2 + \frac{m\omega^2}{2}\hat{\mathbf{X}}^2\big\}$ is not empty. Furthermore, we mentioned earlier that $\mathbf{Spec}\big\{\hat{\mathbf{X}}\big\} = \mathbb{R}$, and we also mentioned that $\mathbf{Spec}\big\{\hat{\mathbf{X}}\big\} = \mathbf{Spec}_c\big\{\hat{\mathbf{X}}\big\}$. We will now shed light on these claims.

First consider the operator $\hat{\mathbf{X}} - \lambda\mathbb{I}$, we will not prove that the range of said operator is dense in $L^2\big(\mathbb{R}\big)$ but this can be shown. Let us act on some arbitrary $\big|\psi\big\rangle$ in the Hilbert space with $\hat{\mathbf{X}}$. The result is $\big(x-\lambda\big)\psi(x)$. We can always choose a $\psi(x) \in L^2\big(\mathbb{R}\big)$ so that for a given $\varepsilon > 0$ we have $\|\big(x-\lambda\big)\psi(x)\|_{L^2(\mathbb{R})} \leq \varepsilon$, e.g. by letting $\psi(x) \in \{\phi_n(x-\lambda)\}_n$ where $\phi_n(x-\lambda)$ is a delta sequence centered at λ we may pick an appropriate n that satisfies the desired bound $\varepsilon > 0$. This, of course, entails that $\big\langle\psi\big|\big(\hat{\mathbf{X}} - \lambda\mathbb{I}\big)\big|\psi\big\rangle$ may be made arbitrarily small, which in turn implies that $\big\langle\psi\big|\big(\hat{\mathbf{X}} - \lambda\mathbb{I}\big)^{-1}\big|\psi\big\rangle$ may be made arbitrarily large; making $\big(\hat{\mathbf{X}} - \lambda\mathbb{I}\big)^{-1}$ an unbounded operator. i.e. even though $\hat{\mathbf{X}}$ has no eigenvalues, it behaves quite singular in some sense. Now, using the fact that the range of $\hat{\mathbf{X}} - \lambda\mathbb{I}$ is dense in $L^2\big(\mathbb{R}\big)$ this operator is basically singular in the entire Hilbert space. Since the latter may be argued for any λ, we conclude that $\mathbf{Spec}_c\big\{\hat{\mathbf{X}}\big\} = \mathbb{R}$. Furthermore, since self-adjoint-operators have real spectrum we conclude that $\mathbf{Spec}\big\{\hat{\mathbf{X}}\big\} = \mathbf{Spec}_c\big\{\hat{\mathbf{X}}\big\}$. Given that both the position operator and the QSHO operator are unbounded operators, one might have suspected some similarities in the nature of their respective spectrum; yet nothing is further from the truth! As our analysis of the position and QSHO operators exemplifies, spectrum type need not be correlated to operator type in general. For the general setting of unbounded and bounded operators, the latter holds true; however, for the more specialized case of trace class and Hibert-Schmidth operators, it turns out that they have only point spectra [3].

Much more can be said about the spectrum of self-adjoint operators and finding the corresponding eigenvectors/ generalized eigenvectors but we will stop here for now. The hope is that the reader gets a sense of the complexity invoked when working with operators acting on infinite-dimensional Hilbert spaces. Not only is finding the spectrum of a given Hamiltonian computationally more daunting in such a case, but in general, it is also very difficult to categorize the spectrum such a Hamiltonian might have. There exist plenty of results in the literature of Schrödinger operators whose aim is to categorize these sorts of operators based on the spectral properties afforded by their potentials; the interested reader on these matters may consult [4] for a phenomenal exposition on this subject.

1.3.1 The Density Operator

Let us take a step back and rewrite the Schrödinger equation in a form that will be more practical for the study of open quantum systems. Starting with a Hilbert space $\mathscr{H}$ and some Hamiltonian $\hat{\mathbf{H}}$ acting in $\mathscr{H}$, we have been interested in the equation (1.1) $i\partial_t\big|\psi_t\big\rangle = \hat{\mathbf{H}}\big|\psi_t\big\rangle$, the solution of which is $e^{-it\hat{\mathbf{H}}}\big|\psi_0\big\rangle = \big|\psi_t\big\rangle$. Let us now call $\hat{\boldsymbol{\rho}}_t := \big|\psi_t\big\rangle\big\langle\psi_t\big|$. Taking the time derivative of this object one gets the following.

$$\partial_t\hat{\boldsymbol{\rho}}_t = \Big(\partial_t|\psi_t\rangle\Big)\langle\psi_t| + |\psi_t\rangle\Big(\partial_t\langle\psi_t|\Big) = -i\hat{\mathbf{H}}|\psi_t\rangle\langle\psi_t| + |\psi_t\rangle\langle\psi_t|i\hat{\mathbf{H}} = \tag{1.29}$$

$$-i\hat{\mathbf{H}}\hat{\boldsymbol{\rho}}_t + \hat{\boldsymbol{\rho}}_t i\hat{\mathbf{H}} = -i\big[\hat{\mathbf{H}},\, \hat{\boldsymbol{\rho}}_t\big]. \tag{1.30}$$

$\partial_t\hat{\boldsymbol{\rho}}_t = -i\big[\hat{\mathbf{H}},\, \hat{\boldsymbol{\rho}}_t\big]$ is called the Liouville-von Neumann equation (LV). The operator $\hat{\boldsymbol{\rho}}_t$ is an instance of what is known as a density operator [9] (chapter 2), also known as a density matrix for the case where $\mathbf{dim}\{\mathscr{H}\} < \infty$.

Definition 1.3.2 (*Density Operator*)

Density operators are positive trace class operators with trace 1. This set forms a convex subset of the relevant Banach algebra that they live in. Assuming that we are working in some *Hilbert* space $\mathscr{H}$, we will denote the space of density operators acting in $\mathscr{H}$ as $\mathcal{S}(\mathscr{H})$. Positive in this context means that given some density operator $\hat{\boldsymbol{\rho}}$, and for any $|\psi\rangle \in \mathscr{H}$, $\langle\psi|\hat{\boldsymbol{\rho}}|\psi\rangle \geq 0$.

For the case of closed systems the density operator starting off in a pure state at time $t = 0$, i.e. some $|\psi\rangle$ in the relevant Hilbert space, has a corresponding time-evolved density operator $\hat{\boldsymbol{\rho}}_t$ which is always a projector. We mentioned the trace earlier when defining trace class operators. The trace is a linear operator which we shall be using frequently, therefore a detailed treatment is in order. We begin with one of the more standard definitions of the trace.

Definition 1.3.3 (Trace)

Let $\mathscr{H}$ be some arbitrary *Hilbert* and assume that $\hat{\boldsymbol{\sigma}} \in \mathcal{B}(\mathscr{H})$. Furthermore, let $\{|\psi_n\rangle\}_n$ be any orthonormal basis of $\mathscr{H}$. Then the trace is defined as follows.

$$Tr\{\hat{\boldsymbol{\sigma}}\} := \sum_n \langle\psi_n|\hat{\boldsymbol{\sigma}}|\psi_n\rangle$$

The value of the trace, assuming that it exists, is independent of the basis chosen.

Assuming that $\hat{\boldsymbol{\rho}},\, \hat{\boldsymbol{\sigma}} \in \mathcal{S}(\mathscr{H})$ for some arbitrary Hilbert space $\mathscr{H}$, and $|\phi\rangle \in \mathscr{H}$, the following list of properties are true.

$$1)\ Tr\big\{a\hat{\boldsymbol{\rho}} + b\hat{\boldsymbol{\sigma}}\big\} = aTr\big\{\hat{\boldsymbol{\rho}}\big\} + bTr\big\{\hat{\boldsymbol{\sigma}}\big\} \tag{1.31}$$

$$2)\ Tr\big\{\hat{\boldsymbol{\rho}}\big\} = \sum_n \lambda_n\big(\hat{\boldsymbol{\rho}}\big)\ (Sum\ over\ eigenvalues) \tag{1.32}$$

$$3)\ Tr\big\{\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\sigma}}\big\} = Tr\{\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\rho}}\} \tag{1.33}$$

$$4)\ Tr\{\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\rho}}\} \leq Tr\{\hat{\boldsymbol{\sigma}}\}Tr\{\hat{\boldsymbol{\rho}}\} \tag{1.34}$$

$$5)\ Tr\{|\phi\rangle\langle\phi|\} = |\langle\phi|\phi\rangle|^2 \tag{1.35}$$

Assuming that we have a quantum system in state $\hat{\boldsymbol{\rho}}$, we may compute expectation values of an arbitrary observable $\hat{\mathbf{A}}$ as follows.

$$6)\ \langle\hat{\mathbf{A}}\rangle_{\hat{\boldsymbol{\rho}}} = Tr\{\hat{\boldsymbol{\rho}}\hat{\mathbf{A}}\}\ (Expectation\ Value\ w.r.t\ state\ \hat{\boldsymbol{\rho}}) \tag{1.36}$$

When a density operator is a projector we will call it a *pure state*, otherwise, it will be a convex mixture of density operators $\hat{\boldsymbol{\sigma}} = \sum_n p_n \hat{\boldsymbol{\sigma}}_n,\ \hat{\boldsymbol{\sigma}}_n$ are density operators and $\sum_n p_n = 1$. Such states will be referred to as *mixed states*. Mixed states of course live in the space $\mathcal{S}(\mathscr{H})$. Owing to their projective properties, any pure state $\hat{\boldsymbol{\sigma}}$ will satisfy the the equality $Tr\{\hat{\boldsymbol{\sigma}}^2\} = Tr\{\hat{\boldsymbol{\sigma}}\}$; on the other hand, for a mixed states $Tr\{\hat{\boldsymbol{\sigma}}^{\mathbf{2}}\} < 1$ [9] (Chapter 2). The map $\gamma(\hat{\mathbf{A}}) := Tr\{\hat{\mathbf{A}}^{\mathbf{2}}\}$is known as the *purity* [9] and it is one of many measures of mixedness for density operators. For any density operator $\hat{\boldsymbol{\rho}}$ acting in a finite-dimensional Hilbert space $\mathscr{H}$, the purity will always be bounded as follows.

$$\frac{1}{\mathbf{dim}\{\mathscr{H}\}} \leq \mathrm{Tr}\left\{\hat{\boldsymbol{\rho}}^{\mathbf{2}}\right\} \leq 1. \tag{1.37}$$

Mixed states have a natural interpretation as a probabilistic ensemble. This comes about physically when the system is known to be in, say, one of the states of a given ensemble $\{p_n, \hat{\boldsymbol{\sigma}}_n\}$ but there is no definite knowledge as to which of these elements it is other than a probability distribution p_n [9].

The concept of mixed states may also be used to quantify the quantumness of a system in the following sense. Consider a quantum system in the pure state $\hat{\boldsymbol{\rho}}_t = |\psi_t\rangle\langle\psi_t|$ at time t, and consider a situation where we are interested in measuring some observable $\hat{\mathbf{X}}$. For simplicity assume that we are working in a finite-dimensional Hilbert space and that the observable $\hat{\mathbf{X}}$ has full rank. We may then diagonalize $\hat{\mathbf{X}}$ and use its eigenvectors $\{|\phi_n\rangle\}_n$ to represent the density operator $\hat{\boldsymbol{\rho}}_t$ as follows.

$$\hat{\boldsymbol{\rho}}_t = \sum_{n,m} \alpha_n(t)\alpha_m^*(t)|\phi_n\rangle\langle\phi_m|. \tag{1.38}$$

where $\alpha_n(t) := \langle\phi_n|\psi_t\rangle$. In the case where $|\psi_t\rangle$ is an eigenvector of $\hat{\mathbf{X}}$, say $|\phi_j\rangle$, (1.38) is equal to $|\phi_j\rangle\langle\phi_j|$ as expected. If we however assume that $|\psi_t\rangle$ is not an eigenvector of $\hat{\mathbf{X}}$, then in such a case (1.38) will have off-diagonal entries in the $\{|\phi_n\rangle\}_n$ representation. This structure will persist so long as the system is described by a superposition state. To see this note that (1.38) is just

$$(1.38) = \left(\sum_n \alpha_n(t)|\phi_n\rangle\right)\left(\sum_m \langle\phi_m|\alpha_m^*(t)\right) \tag{1.39}$$

which is a diad of superposition states. The off-diagonal terms of the density are therefore an inherently quantum feature, making the case where the off-diagonal entries are zero minimally quantum. The off-diagonal terms of the matrix will not be zero in general. In particular, under unitary evolution (1.38) will always have off-diagonal entries. In cases, which we will be discussing shortly, where the dynamics are not generated by a unitary group (1.2), the off-diagonal entries $\alpha_n(t)\alpha_m^*(t)$ will decay with respect to t. For large t we would therefore have

$$\sum_{n,m} \alpha_n(t)\alpha_m^*(t)\big|\phi_n\big\rangle\big\langle\phi_m\big| \approx \sum_n |\alpha_n(t)|^2\big|\phi_n\big\rangle\big\langle\phi_n\big| \tag{1.40}$$

which is a mixed state of a family of orthogonal projectors. Such a state represents a classical probability distribution since the different states $\big|\phi_n\big\rangle$ are distinguishable amongst one another. Note that there is no way to represent the right-hand side of (1.40) as a pure state. The mixture (1.40) is a minimally quantum one with respect to the observable $\hat{\mathbf{X}}$. Mixtures need not be minimally quantum in general, specific types of dynamics are required to achieve this. We have been using the term quantumness in a vague sense up until this point, now that we have the tools to formalize what we wish to convey with such a word we present a mathematical definition for *quantumness*.

Definition 1.3.4 (Quantumness)

Let $\mathscr{H}$ be an arbitrary *Hilbert* space and consider a general quantum state $\hat{\boldsymbol{\rho}} \in \mathcal{S}\big(\mathscr{H}\big)$. A general quantum sate $\hat{\boldsymbol{\rho}}$ may always be written in the convex linear combination $\hat{\boldsymbol{\rho}} = \sum_n p_n \hat{\boldsymbol{\rho}}_n$. The quantumness of a density operator $\hat{\boldsymbol{\rho}}$ will be defined as

$$\mathcal{Q}\big(\hat{\boldsymbol{\rho}}\big) := \begin{cases} \left|\frac{1}{2}\sum_n \sum_{m;m\neq n} p_n p_m Tr\big\{\hat{\boldsymbol{\rho}}_n\hat{\boldsymbol{\rho}}_m\big\}\right| & (Mixed\ \hat{\boldsymbol{\rho}}) \\ 1 & (pure\ \hat{\boldsymbol{\rho}}) \end{cases} \tag{1.41}$$

which is a representation-independent quantity. Indeed $0 \leq \mathcal{Q}\big(\hat{\boldsymbol{\rho}}\big) \leq 1$.

Quantumness may also be defined in terms of the purity γ.i.e.

$$\mathcal{Q}\big(\hat{\boldsymbol{\rho}}\big) := \begin{cases} \left|\gamma\big(\hat{\boldsymbol{\rho}}\big) - \sum_n p_n \gamma\big(\hat{\boldsymbol{\rho}}_n\big)\right| & (Mixed\ \hat{\boldsymbol{\rho}}) \\ 1 & (pure\ \hat{\boldsymbol{\rho}}) \end{cases} \tag{1.42}$$

To make sense of the value of $\mathcal{Q}(\hat{\boldsymbol{\rho}}) = 1$ for pure states consider the mixed state $\hat{\boldsymbol{\rho}}_\varepsilon = \frac{1}{2}\hat{\boldsymbol{\sigma}}_1 + \frac{1}{2}\hat{\boldsymbol{\sigma}}_{2,\varepsilon}$, with $\hat{\boldsymbol{\sigma}}_1$ a pure state and $\hat{\boldsymbol{\sigma}}_{2,\varepsilon} := \frac{\hat{\boldsymbol{\sigma}}_1 + \varepsilon\hat{\boldsymbol{\eta}}}{1+\varepsilon}$ ($\hat{\boldsymbol{\eta}}$ an arbitrary density operator). Then

$$\lim_{\varepsilon\to 0} \mathcal{Q}\big(\hat{\boldsymbol{\rho}}_\varepsilon\big) = \lim_{\varepsilon\to 0} \left|Tr\big\{\hat{\boldsymbol{\sigma}}_1\hat{\boldsymbol{\sigma}}_{2,\varepsilon}\big\}\right| = \lim_{\varepsilon\to 0} \frac{1}{1+\varepsilon}\left|\Big(Tr\{\hat{\boldsymbol{\sigma}}_1^2\} + \varepsilon Tr\{\hat{\boldsymbol{\sigma}}_1\hat{\boldsymbol{\eta}}\}\Big)\right| = \tag{1.43}$$

$$Tr\{\hat{\boldsymbol{\sigma}}_1^2\} = 1 \tag{1.44}$$

This means that pure states may be approximated by mixed states with quantumness arbitrarily close to one. This is in line with our intuition of pure states being the most quantum-like. Now, consider the right-hand side of (1.40) again. In this case

$$\mathcal{Q}\left(\sum_n |\alpha_n(t)|^2 \big|\phi_n\big\rangle\big\langle\phi_n\big|\right) = \frac{1}{2}\Big|\sum_n \sum_{m;m\neq n} p_n p_m \big|\big\langle\phi_n\big|\phi_m\big\rangle\big|^2\Big| = \tag{1.45}$$

$$\frac{1}{2}\Big|\sum_n \sum_{m;m\neq n} p_n p_m \delta_{n,m}\Big| = 0 \tag{1.46}$$

which is minimal quantumness as per Definition 1.3.4. Mixed states allow us to estimate the level of quantumness in a system to the extent that we are able to express the state in question as a mixture, which is always possible. Quantumness may be used as a measure of *decoherence* [12] which is the theory concerned with studying the transition of quantum states to classical probability states as noted in (1.40). The measure we have defined in Definition 1.3.4 allows us to study the latter but in an inverted sense. i.e. minimal quantumness will correspond with maximal decoherence and vice-versa. We now present a mathematical definition for decoherence.

Definition 1.3.5 (Decoherence Measure)

Let $\mathscr{H}$ be an arbitrary *Hilbert* space and consider a general quantum state $\hat{\boldsymbol{\rho}} \in \mathcal{S}(\mathscr{H})$. A general quantum sate $\hat{\boldsymbol{\rho}}$ may always be written in the convex linear combination $\hat{\boldsymbol{\rho}} = \sum_n p_n \hat{\boldsymbol{\rho}}_n$. The decoherence measure of a density operator $\hat{\boldsymbol{\rho}}$ will be defined as

$$\mathcal{DK}(\hat{\boldsymbol{\rho}}) := 1 - \mathcal{Q}(\hat{\boldsymbol{\rho}}) \tag{1.47}$$

Indeed $0 \leq \mathcal{DK}(\hat{\boldsymbol{\rho}}) \leq 1$. Maximum and minimum decoherences occur when $\mathcal{DK}(\hat{\boldsymbol{\rho}}) = 1$ and $\mathcal{DK}(\hat{\boldsymbol{\rho}}) = 0$ respectively.

1.3.2 Trace for the Infinite-Dimensional $\mathscr{H}$ Case

Let $\mathscr{H}$ be some infinite dimensional Hilbert space and let $\hat{\boldsymbol{K}} \in \mathcal{S}(\mathscr{H})$. Therefore

$$Tr\{\hat{\boldsymbol{K}}\} = \sum_{n=0}^{\infty} \big\langle\phi_n\big|\hat{\boldsymbol{K}}\big|\phi_n\big\rangle < \infty \tag{1.48}$$

since density operators are trace class, where $\overline{\{|\phi_n\rangle\}_{n=0}^{\infty}} = \mathscr{H}$. Recall that trace-class operators have only point-spectrum (eigenvalues). Therefore, if the set $\{|\phi_n\rangle\}_{n=0}^{\infty}$ is taken to be the set of eigenvectors of $\hat{\boldsymbol{K}}$, then

$$Tr\{\hat{\boldsymbol{K}}\} = \sum_{n=0}^{\infty} \lambda_n(\hat{\boldsymbol{K}}). \tag{1.49}$$

This sum produces no issues since the spectrum of compact self-adjoint operators is always absolutely summable.

As we discussed in the previous subsection, calculating the spectrum of an arbitrary trace-class operator is in general an arduous task. Here we would need an infinite amount of eigenvalues and eigenvectors in order to calculate the trace. In some instances, however, the trace class operator $\hat{\boldsymbol{K}}$ will be an integral operator, and the theory of traces for integral operators has some useful results that we may tap into. The following is a direct consequence of *Mercer*'s theorem from functional analysis [30].

Theorem 1.3.1 (Corollary to Mercer's Theorem)

Suppose $K(x, y)$ is a continuous, symmetric positive-definite kernel that is compactly supported; define

$$\hat{\boldsymbol{K}} : \mathscr{H} \to \mathscr{H} := |\psi\rangle \to \int_{\mathbb{R}} \int_{\mathbb{R}} K(x, y)|x\rangle\langle y|\psi\rangle dx dy \tag{1.50}$$

then

$$Tr\{\hat{\boldsymbol{K}}\} = \int_{\mathbb{R}} K(x, x) dx \tag{1.51}$$

The way integral operators come about in closed quantum systems may be exemplified by the following. Consider the case where $\mathscr{H} = L^2(\mathbb{R})$, and let $|\psi\rangle\langle\psi| \in \mathcal{S}(L^2(\mathbb{R}))$. Let us now act on both sides of $|\psi\rangle\langle\psi|$ with the identity (where we use some conventions from physics [6] Chapter 6) .

$$\Big(\int |x\rangle\langle x| dx \Big) |\psi\rangle\langle\psi| \Big(\int |x\rangle\langle x| dx \Big) = \tag{1.52}$$

$$\int \int \psi^*(x)\psi(y)|x\rangle\langle y| dx dy \tag{1.53}$$

In function notation such an operator acts on an arbitrary function $f(x) \in L^2(\mathbb{R})$ as follows.

$$|\psi\rangle\langle\psi| : f(x) \to \psi(x) \int_{\mathbb{R}} \psi^*(y) f(y) dy. \tag{1.54}$$

In physics, assuming that the states $\psi(x)$ are compactly supported and continuous is usually physically reasonable and more faithful to reality than otherwise. Hence, Theorem 1.3.1 is truly all we need for the physical case. However, Theorem 1.3.1 was generalized by Brislawn to include a wider family of kernels $K(x, y)$; this includes any Hilbert-Schmidt Kernel [3], which will be our primary focus. This means that we may work in the more general setting allowed by Brislawn's results [47] and rest at ease knowing that such generalities include the physical setting encompassed by Theorem 1.3.1. Rather than cite all of the relevant papers that Brislawn published, the interested reader is recommended to consult [47](Addenda D) for a succinct discussion and detailed list of references.

1.4 Open Quantum Systems and the Partial Trace

The point of departure from closed to open quantum systems takes place when one assumes that the system of interest, which we will refer to as S, is interacting with another system/ other systems which we will call the E^k for the kth systems. We use the letter E to emphasize the dichotomy between <u>S</u>ystem and <u>E</u>nvironments. The total dynamics of the system S and the environments E^k will not be our focus, but rather the local dynamics pertaining to S. To treat compound quantum systems, one must construct bigger Hilbert spaces. i.e. let $\mathscr{H}_S$ and $\mathscr{H}_{E^k}$ all be arbitrary Hilbert spaces and let $\hat{\boldsymbol{\rho}}_S \in \mathcal{S}(\mathscr{H}_S)$, $\hat{\boldsymbol{\rho}}^{E^k} \in \mathcal{S}(\mathscr{H}_{E^k})$ (This superscript and subscript convention will make things easier to organize later). Furthermore assume that $\hat{\mathbf{H}}_{tot}$ is a Hamiltonian acting in $\mathscr{H}_S \otimes \bigotimes_k \mathscr{H}_{E^k}$.The Liouville-von Neumann equation (1.30) then has the following solution.

$$e^{-it\hat{\mathbf{H}}_{tot}}\Big(\hat{\boldsymbol{\rho}}_S \bigotimes_k \hat{\boldsymbol{\rho}}^{E^k}\Big)e^{it\hat{\mathbf{H}}_{tot}} \tag{1.55}$$

In principle, the state (1.55) contains all information about the state of the system S and all environments E^k after time evolution. In the case where some scientist conducting experiments on S has no means by which to measure the properties of the E^k, such a state (1.55) includes more than can be known. The state of the object the scientist is observing might live only in $\mathscr{H}_S$, but due to the lack of knowledge regarding the correlations of S to the E^k the state of S will now evolve in a non-unitary fashion since information and energy and perhaps other things might be traded between S and the E^k as time evolves. If the E^k are large enough, such an evolution may become irreversible!

How does one isolate the local dynamics pertaining to the system S? The answer is simple, we take a trace! But not over all of $\mathscr{S}_1\Big(\mathscr{H}_S \otimes \bigotimes_k \mathscr{H}_{E^k}\Big)$, we trace only over the stuff we cannot physically track, in this case, this is $\mathscr{S}_1\big(\bigotimes_k \mathscr{H}_{E^k}\big)$. This brings us to the definition of the partial trace. We write the definition in a way that is suggestive of the models we shall be working on.

Definition 1.4.1 (Partial Trace)

Let $\mathscr{H}_S \otimes \bigotimes_k \mathscr{H}_{E^k}$ be some arbitrary tensor product *Hilbert* space, and assume that $\hat{\boldsymbol{\sigma}} \in \mathcal{B}\Big(\mathscr{H}_S \otimes \bigotimes_k \mathscr{H}_{E^k}\Big)$. Furthermore, let $\big\{|\phi_n^{E^k}\rangle\big\}_n$ be any orthonormal basis of $\mathscr{H}_{E^k}$. Then the partial trace over $\mathcal{B}(\mathscr{H}_{E^l})$ is the linear map

$$Tr_{E^k} : \mathcal{B}\Big(\mathscr{H}_S \otimes \bigotimes_k \mathscr{H}_{E^k}\Big) \to \mathscr{L}\Big(\mathscr{H}_S \otimes \bigotimes_{k;k\neq l} \mathscr{H}_{E^k}\Big)$$

defined as follows

$$Tr_{E^l}\{\hat{\boldsymbol{\sigma}}\} = \sum_n \langle \phi_n^{E^l}|\hat{\boldsymbol{\sigma}}|\psi_n^{E^l}\rangle$$

where $\mathscr{L}(\mathscr{H})$ refers to the space of linear operators acting in $\mathscr{H}$. The value of the partial trace is independent of the basis chosen.

If we specialize the partial trace to only the space $\mathcal{S}\Big(\mathscr{H}_S \otimes \bigotimes_k \mathscr{H}_{E^k}\Big)$, then all of the partial traces will exist and will be once again density operators.

Why the partial trace?

To assure ourselves that the partial trace is the appropriate mapping to use in the deduction of the local dynamics consider an arbitrary observable $\hat{\mathbf{A}}_S$ acting in $\mathscr{H}_S$. There is a natural embedding of such an observable that extends it to the space of observables acting in $\mathscr{H}_S \otimes \bigotimes_{k=1}^{N_E} \mathscr{H}_{E^k}$.

$$\hat{\mathbf{A}}_S \to \hat{\mathbf{A}}_S \otimes \bigotimes_{k=1}^{N_E} \mathbb{I}_{E^k}, \tag{1.56}$$

where $\mathbb{I}_{E^k}$ is the identity operator of $\mathcal{B}(\mathscr{H}_E^k)$. Viewing $A_S \otimes \bigotimes_{k=1}^{N_E} I_{E^k}$ as an observable acting in the total Hilbert space $\mathscr{H}_S \otimes \bigotimes_{k=1}^{N_E} \mathscr{H}_{E^k}$ one can obtain the expectation value using the total system's density operator using (1.36) as follows

$$\left\langle \hat{\mathbf{A}}_S \otimes \bigotimes_{k=1}^{N_E} \mathbb{I}_{E^k} \right\rangle_{\hat{\boldsymbol{\rho}}_{SE}} = Tr_S\Big\{Tr_{E^1}\Big\{T_{E^2}\Big\{...\Big\{Tr_{E^{N_E}}\Big\{\hat{\boldsymbol{\rho}}_{SE}\Big(\hat{\mathbf{A}}_S \otimes \bigotimes_{k=1}^{N_E} \mathbb{I}_{E^k}\Big)\Big\}\Big\}\Big\}\Big\}\Big\} \tag{1.57}$$

In the usual closed quantum systems setting there are no correlations assumed between S and any E^k in the universe. The total state $\hat{\boldsymbol{\rho}}_{SE}$ must therefore be a product state; for the readers knowledgeable of quantum entanglement we briefly comment that an uncorrelated state is forcibly unentangled, quantum correlations being more general than entanglement (see [45] chapter 4). Hence, $\hat{\boldsymbol{\rho}}_{SE}$ has the form $\hat{\boldsymbol{\rho}}_S \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{E^k}$, with $\hat{\boldsymbol{\rho}}_S \in \mathcal{B}(\mathscr{H}_S)$, $\hat{\boldsymbol{\rho}}^{E^k} \in \mathcal{B}(\mathscr{H}_{E^k})$. We now compute (1.57) to obtain

$$Tr_S\Big\{Tr_{E^1}\Big\{T_{E^2}\Big\{...\Big\{Tr_{E^{N_E}}\Big\{\hat{\boldsymbol{\rho}}_S \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{E^k}\Big(\hat{\mathbf{A}}_S \otimes \bigotimes_{k=1}^{N_E} \mathbb{I}_{E^k}\Big)\Big\}\Big\}\Big\}\Big\}\Big\} = \tag{1.58}$$

$$Tr_S\Big\{Tr_{E^1}\Big\{T_{E^2}\Big\{...\Big\{Tr_{E^{N_E}}\Big\{\hat{\boldsymbol{\rho}}_S \hat{\mathbf{A}}_S \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{E^k}\Big\}\Big\}\Big\}\Big\}\Big\} = \tag{1.59}$$

$$Tr_S\Big\{\hat{\boldsymbol{\rho}}_S \hat{\mathbf{A}}_S\Big\} \prod_{k=1}^{N_E} Tr_{E^k}\{\hat{\boldsymbol{\rho}}_{E^k}\} = Tr_S\Big\{\hat{\boldsymbol{\rho}}_S \hat{\mathbf{A}}_S\Big\} = \tag{1.60}$$

$$\Big\langle \hat{\boldsymbol{\rho}}_S \hat{\mathbf{A}}_S \Big\rangle_{\hat{\boldsymbol{\rho}}_S} \tag{1.61}$$

We, therefore, conclude that

$$\left\langle \hat{\mathbf{A}}_S \otimes \bigotimes_{k=1}^{N_E} \mathbb{I}_{E^k} \right\rangle_{\hat{\boldsymbol{\rho}}_{SE}} = \Big\langle \hat{\boldsymbol{\rho}}_S \hat{\mathbf{A}}_S \Big\rangle_{\hat{\boldsymbol{\rho}}_S} \tag{1.62}$$

Meaning that the information needed to compute the statistical properties of some observable in S are completely contained in the reduced density operator

$$Tr_{E^1}\Big\{T_{E^2}\Big\{...\Big\{Tr_{E^{N_E}}\Big\{\hat{\boldsymbol{\rho}}_S \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{E^k}\Big\}\Big\}\Big\}\Big\} = \hat{\boldsymbol{\rho}}_S \tag{1.63}$$

In this discussion, we have assumed a tensor product structure for the environmental degrees of freedom, this was not a technical assumption, however. We may arrive at the same conclusion (1.62) without the latter assumption via an identical approach.

Non-dynamical example

To understand the effects of the environments E^k on measurements being conducted in S, let us take the environment to be a single E in the state $\hat{\boldsymbol{\rho}}^E \in \mathcal{S}\big(\mathscr{H}_E\big)$, where $\mathscr{H}_E$ is an arbitrary Hilbert space. Let us assume that the sate $\hat{\boldsymbol{\rho}}^E$ is diagonal with respect to some basis $\big\{\big|E_i\big\rangle\big\}_i$. Finally, let us assume that the state of S is pure, namely $\big|\psi\big\rangle\big\langle\psi\big|$, for some vector $\big|\psi\big\rangle \in \mathscr{H}_S$ (an arbitrary Hilbert space). Consider an observable $\hat{\mathbf{X}}$ with countable $\mathbf{Spec}\big\{\hat{\mathbf{X}}\big\}$ which is all point spectrum, and associated eigenvectors $\big\{\big|\eta_i\big\rangle\big\}_i$ that span $\mathscr{H}_S$; for the case where $\mathscr{H}_S$ span will of course refer to the closure of the span of $\big\{\big|\eta_i\big\rangle\big\}_i$. We may represent $\big|\psi\big\rangle\big\langle\psi\big|$ in such a basis as $\sum_i \sum_j \alpha_i\alpha_j^*\big|\eta_i\big\rangle\big\langle\eta_j\big|$; where $\alpha_i := \big\langle\eta_i\big|\psi\big\rangle$. Furthermore, consider a state $\hat{\boldsymbol{\rho}}_{SE} \in \mathcal{S}\big(\mathscr{H}_S \otimes \mathscr{H}_E\big)$ with the following representation.

$$\hat{\boldsymbol{\rho}}_{SE} = \sum_{i,j} \alpha_i\alpha_j^*|\eta_i\rangle\langle\eta_j\big| \otimes \big|E_i\big\rangle\big\langle E_j\big|. \tag{1.64}$$

Such a state is non-separable, i.e. it may not be written as a tensor product $\hat{\boldsymbol{\rho}}_S \otimes \hat{\boldsymbol{\rho}}_E$ where $\hat{\boldsymbol{\rho}}_S$ and $\hat{\boldsymbol{\rho}}_E$ are respectively elements of $\mathcal{S}\big(\mathscr{H}_S\big)$ and $\mathcal{S}\big(\mathscr{H}_E\big)$; such a state (1.64) therefore describes correlations between S and E. As before, given some local observable $\hat{\mathbf{A}}_S$ of $\mathscr{H}_S$, we would like to compute its statistics; starting with the expectation value. The density operator we need is the reduced state of (1.64). i.e.

$$Tr_E\big\{\hat{\boldsymbol{\rho}}_{SE}\big\} = Tr_E\Big\{\sum_{i,j} \alpha_i\alpha_j^*|\eta_i\rangle\langle\eta_j\big| \otimes \big|E_i\big\rangle\big\langle E_j\big|\Big\} = \tag{1.65}$$

$$\sum_{i,j} \alpha_i\alpha_j^* Tr_E\Big\{\big|E_i\big\rangle\big\langle E_j\big|\Big\}|\eta_i\rangle\langle\eta_j\big| = \sum_{i,j} \alpha_i\alpha_j^*\delta_{ij}|\eta_i\rangle\langle\eta_j\big| = \tag{1.66}$$

$$\sum_i |\alpha_i|^2\big|\eta_i\rangle\langle\eta_i\big| \tag{1.67}$$

This is a noteworthy result for two reasons. First, the resulting density operator is diagonal. Taking the partial trace over the perfectly distinguishable environmental degrees of freedom has induced minimal quantumness (Definition 1.3.4) and therefore maximal decoherence (Definition 1.3.5). Second, at the risk of being redundant, one should remark that the resulting density operator is a mixed state in spite of $\big|\psi\big\rangle\big\langle\psi\big|$ having been pure to begin with. It is the correlations with E and our ignorance of the information of the state of E that induces maximal decoherence and leaves us with a state that is de facto a classical probability distribution.

1.5 Environmentally Induced Non-Unitarity

We have seen that the correlations between some quantum system S and its environmentsE^k are the source of decoherence. The imminent question is now, how do such correlations arise? To shed light on this question consider once again a total Hilbert space $\mathscr{H}_S \otimes \mathscr{H}_E$ and let $\hat{\mathbf{H}}_{tot}$ be some Hamiltonian acting in such a Hilbert space. Since we are only interested in the creation of correlations, as opposed to the time-evolution of pre-existing correlations, we will narrow our attention to initial states $\hat{\boldsymbol{\rho}}_{SE_0} \in \mathcal{S}\big(\mathscr{H}_S \otimes \mathscr{H}_E\big)$ that are separable. i.e. $\hat{\boldsymbol{\rho}}_{SE_0} = \hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0}$. The time-evolution operator obtained from solving the LV equation (1.30) in such a case is $\hat{\mathbf{U}}_t = e^{-it\hat{\mathbf{H}}_{tot}}$. The local time evolution of the open quantum system S is hence

$$\hat{\boldsymbol{\rho}}_{S_t} = Tr_E\big\{\hat{\mathbf{U}}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\hat{\mathbf{U}}_t^{\dagger}\big\}. \tag{1.68}$$

Now, assume that the environment E is in the diagonal state $\hat{\boldsymbol{\rho}}^{E_0} = \sum_i \alpha_i|E_i\rangle\langle E_i|$; this assumption does not affect generality since one can always use the invariance of the trace under unitary transformations in order to diagonalize the state $\hat{\boldsymbol{\rho}}^{E_0}$, such a diagonalization will, of course, change the representation of $\hat{\mathbf{H}}_{tot}$. The state (1.68) can now be expanded as seen below.

$$Tr_E\Big\{\hat{\mathbf{U}}_t\Big(\hat{\boldsymbol{\rho}}_{S_0}\otimes\sum_i\alpha_i|E_i\rangle\langle E_i|\Big)\hat{\mathbf{U}}_t^{\dagger}\Big\} \tag{1.69}$$

$$=\sum_i\sum_j\sqrt{\alpha_i}\big\langle E_j\big|\hat{\mathbf{U}}_t\Big(\hat{\boldsymbol{\rho}}_{S_0}\otimes|E_i\rangle\langle E_i|\Big)\hat{\mathbf{U}}_t^{\dagger}\big|E_j\big\rangle\sqrt{\alpha_i}= \tag{1.70}$$

$$=\sum_i\sum_j\sum_k\sum_l\sqrt{\alpha_i}\big\langle E_j\big|\hat{\mathbf{U}}_t\big|E_k\big\rangle\big\langle E_k\big|\Big(\hat{\boldsymbol{\rho}}_{S_0}\otimes|E_i\rangle\langle E_i|\Big)\big|E_l\big\rangle\big\langle E_l\big|\hat{\mathbf{U}}_t^{\dagger}\big|E_j\big\rangle\sqrt{\alpha_i}= \tag{1.71}$$

$$\sum_i\sum_j\sum_k\sum_l\sqrt{\alpha_i}\big\langle E_j\big|\hat{\mathbf{U}}_t\big|E_k\big\rangle\Big(\hat{\boldsymbol{\rho}}_{S_0}\big\langle E_k\big|E_i\big\rangle\big\langle E_i\big|E_l\big\rangle\Big)\big\langle E_l\big|\hat{\mathbf{U}}_t^{\dagger}\big|E_j\big\rangle\sqrt{\alpha_i}= \tag{1.72}$$

$$\sum_i\sum_j\sum_k\sum_l\sqrt{\alpha_i}\big\langle E_j\big|\hat{\mathbf{U}}_t\big|E_k\big\rangle\Big(\hat{\boldsymbol{\rho}}_{S_0}\delta_{ki}\delta_{il}\Big)\big\langle E_l\big|\hat{\mathbf{U}}_t^{\dagger}\big|E_j\big\rangle\sqrt{\alpha_i}= \tag{1.73}$$

$$\sum_i\sum_j\sqrt{\alpha_i}\big\langle E_j\big|\hat{\mathbf{U}}_t\big|E_i\big\rangle\hat{\boldsymbol{\rho}}_{S_0}\big\langle E_i\big|\hat{\mathbf{U}}_t^{\dagger}\big|E_j\big\rangle\sqrt{\alpha_i} \tag{1.74}$$

The operators

$$\hat{\mathbf{M}}_{ij,t} := \sqrt{\alpha_i}\big\langle E_j\big|\hat{\mathbf{U}}_t\big|E_i\big\rangle \tag{1.75}$$

have some interesting properties. The first is the resolution of identity in the following sense.

$$\sum_i\sum_j\hat{\mathbf{M}}_{ij,t}^{\dagger}\hat{\mathbf{M}}_{ij,t} = \mathbb{I}_S \tag{1.76}$$

Proof.

$$\sum_i\sum_j\hat{\mathbf{M}}_{ij,t}^{\dagger}\hat{\mathbf{M}}_{ij,t} = \sum_i\alpha_i\sum_j\big\langle E_i\big|\hat{\mathbf{U}}_t^{\dagger}\big|E_j\big\rangle\big\langle E_j\big|\hat{\mathbf{U}}_t\big|E_i\big\rangle = \tag{1.77}$$

$$\sum_i\alpha_i\big\langle E_i\big|\hat{\mathbf{U}}_t^{\dagger}\mathbb{I}_S\otimes\Big(\sum_j\big|E_j\big\rangle\big\langle E_j\big|\Big)\hat{\mathbf{U}}_t\big|E_i\big\rangle = \tag{1.78}$$

$$\sum_i\alpha_i\big\langle E_i\big|\hat{\mathbf{U}}_t^{\dagger}\mathbb{I}_S\otimes\mathbb{I}_E\hat{\mathbf{U}}_t\big|E_i\big\rangle \tag{1.79}$$

$$\sum_i\alpha_i\big\langle E_i\big|\mathbb{I}_S\otimes\mathbb{I}_E\big|E_i\big\rangle = \mathbb{I}_S\sum_i\alpha_i = \mathbb{I}_S \tag{1.80}$$

□

The second interesting property is that the operators $\hat{\mathbf{M}}_{ij}(t)$ will, in general, generate non-unitary evolution; with the exception of very special $\hat{\mathbf{H}}_{tot}$, these maps will induce decoherence. To see this, let us specialize the above to a simple case where the Hamiltonian $\hat{\mathbf{H}}_{tot} = \hat{\mathbf{A}}_S \otimes \sum_i E_i \big|E_i\big\rangle\big\langle E_i\big|$. Where $\hat{\mathbf{A}}_S$ is an observable acting in $\mathscr{H}_S$. With such a Hamiltonian it can be easily shown that

$$\hat{\mathbf{M}}_{ij,t} = \sqrt{\alpha}_i e^{-itE_i\hat{\mathbf{A}}_S}\delta_{ij} \tag{1.81}$$

and hence

$$(1.74) = \sum_i \sum_j \alpha_i e^{-itE_i\hat{\mathbf{A}}_S}\delta_{ij}\hat{\boldsymbol{\rho}}_{S_0}e^{itE_i\hat{\mathbf{A}}_S}\delta_{ji} = \tag{1.82}$$

$$\sum_i \alpha_i e^{-itE_i\hat{\mathbf{A}}_S}\hat{\boldsymbol{\rho}}_{S_0}e^{itE_i\hat{\mathbf{A}}_S}. \tag{1.83}$$

The density operator (1.83) is mixed for all t $\big($ unless $\big[\hat{\mathbf{A}}_S,\ \hat{\boldsymbol{\rho}}_S(0)\big] = 0\big)$. This means that the purity of $\hat{\boldsymbol{\rho}}_{S_0}$ will not be preserved and thus the non-unitarity dynamics. For all t, the operators $\big\{\hat{\mathbf{M}}_{ij,t}\big\}_{ij}$ are an instance of what is referred to as a family of *Kraus* operators in the literature of quantum information and quantum computation[9] [45]. Non-unitary evolution will always be generated by a family of Kraus operators. The maps generated by a family of Kraus operators are what is known as a *Quantum Map* in the theory of quantum information and quantum computation [9] [45], a concept that we will develop further in the following chapter. The hurdle that comes from this approach to the modeling of open quantum systems is two-fold. Firstly, one must compute the inner products $\hat{\mathbf{M}}_{ij,t} := \big\langle E_j\big|\hat{\mathbf{U}}_t\big|E_i\big\rangle$. Secondly, assuming that the explicit nature of the map $\hat{\mathbf{M}}_{ij,t}$ is known for all ij, it will remain a difficult task to understand how these operators act on $\hat{\boldsymbol{\rho}}_{S_0}$ from the left and from the right. Both of these aforementioned hurdles will require us to understand the spectral decomposition of the operator $\hat{\mathbf{H}}_{tot}$ in order to understand the explicit nature of $\hat{\mathbf{U}}_t$; a task that we concluded to be in general intractable earlier in this chapter in sections (1.2) and (1.3). All hope is not lost, however, formidable estimation techniques may be implemented in order to tame these hurdles. For the cases where the environment E is very large compared to the systems S, one may implement the so-called *Born-Markov* approximations [17][12][45] that yield a relatively wieldy equation called the GKLS (after its creators Gorini–Kossakowski–Sudarshan–Lindblad) [45] that generalizes the LV equation for the case of the non-unitary dynamic generated by a semigroup. Such an equation allows an indirect estimation of the associated Kraus operators. We present the GKSL equation here, although the models pertaining to the main results of this work will not require us to utilize GKSL equation in any practical sense other than to exhibit some physical examples of non-unitarity in the next section.

Let the total Hamiltonian $\hat{\mathbf{H}}_{tot}$ have the following structure.

$$\hat{\mathbf{H}}_{tot} = \hat{\mathbf{H}}_S + \hat{\mathbf{H}}_E + \hat{\mathbf{H}}_I \tag{1.84}$$

i.e. it will be a linear combination of the *self-Hamiltonians* of S and E respectively as well as an interaction term $\hat{\mathbf{H}}_I$. Under necessary assumptions required by the GKSL [17] framework one may transition from (1.74) to the following; i.e. (1.74) is a solution to an equation of the following form.

$$\partial_t\hat{\boldsymbol{\rho}}_{S_t} = -i\big[\hat{\mathbf{H}}_S, \hat{\boldsymbol{\rho}}_{S_t}\big] + \sum_i \gamma_i\Big(\hat{\mathbf{L}}_i\hat{\boldsymbol{\rho}}_{S_t}\hat{\mathbf{L}}_i^\dagger - \frac{1}{2}\Big\{\hat{\mathbf{L}}_i^\dagger\hat{\mathbf{L}}_i, \hat{\boldsymbol{\rho}}_{S_t}\Big\}\Big) \tag{1.85}$$

This is the GKSL equation, the operators $\hat{\mathbf{L}}_i$ are referred to as *collapse operators* with respective rates γ_i. The elements involving the collapse operators on the right-hand side of (1.85) will constitute the non-unitary dynamics. These collapse operators will induce dissipation and decoherence. The Liouivillian term$-i\left[\hat{\mathbf{H}}_S, \partial_t \hat{\boldsymbol{\rho}}_{S_t}\right]$ will induce the unitary self-dynamics of the system S. Equation (1.85) was originally developed for the case of bounded collapse operators in [48] but recently it has been formalized on equal theoretical grounds for the case of unbounded collapse operators in [49]. In the next section, we will show some physical examples of non-unitary evolution, two of which will arise from solving the appropriate GKSL equation.

1.6 Physical Examples of Open Quantum Systems

Quantum mechanics is a theory of matter which is more fundamental than the classical theories afforded by Newtonian dynamics and Maxwell's equations. Quantum mechanics is therefore the correct theory to describe anything around us. In this sense, everything is a Quantum Open system because everything we deemed to be a system will sit within a larger system. If the latter were not to be the case we would inevitably have to face the Heisenberg-Cut dilemma [34] which asks "where does one draw the boundary between the classical and the quantum?". Classical beings nevertheless exist in a realm where so-called quantum effects may be negligible and classical mechanics is enough to aid us in understanding our environment. However, recent interest in technology which is on the Nanoscale has functioned as the impetus of a deeper interest regarding the non-unitary (open systems) dynamics of quantum mechanical systems. Some Quantum computers constructed by IBM [35], for example, are built from tiny Quantum circuits which are highly susceptible to minuscule disturbances coming from its entourage. In order to truly understand these Quantum circuits and their dynamical properties, a strong understanding of the non-unitary evolution must be taken into account and this requires us to consider the quantum system as open. The latter is also true for any system; as the size of the system becomes smaller the necessity to include interaction terms between the system and its environment becomes greater. We will give three examples of open quantum systems in this section. The first will be the spontaneous emission of a two-level atom. This model exemplifies both dissipative and decoherence effects via the study of an atom, initially in an excited state, interacting with the vacuum. One expects that owing to the vastness of the vacuum compared to the smallness of the atom, the atom will emit a photon and lose its energy to the vacuum. Such a physical system therefore cannot be modeled via the unitary evolution afforded by the LV equation. We will therefore make use of the GKSL equation appropriate for this system in order to study dynamics that are more physically grounded. Next, we will look at another one of the canonical models of decoherence, namely scattering decoherence. This model will be used to exemplify the fact that decoherence need not be accompanied by dissipation whilst dissipation is always accompanied by decoherence. The model will consist of a mesoscopic-sized sphere being bombarded by a field of monochromatic light. We will see that under a very short time scale, virtually full decoherence takes place. The last model we will present will be a monitoring model akin to the main item of study in this thesis (see Chapters 5 and 6). Here we will study a multipartite/multifaceted (we will use both of these terms interchangeably) open quantum system model. Up until now, we have considered S to be one entity;

the formalism we have developed nevertheless allows for $\hat{\boldsymbol{\rho}}_{S_0}$ to be a separable tensor product state at $t = 0$ (multipartite/multifaceted). We may also evolve this multipartite state non-unitarily by considering dynamics between this multipartite system S's state and some environments E^k.

1.6.1 Spontaneous Emission

Consider a two-level atom coupled to a bath in the vacuum states. Such a system may be modeled with $\mathscr{H}_S = \mathbb{C}^2$ and $\mathscr{H}_E = L^2(\mathbb{R})$ for the system S and the environment E respectively. We will let the total Hamiltonian modeling the dynamics be that which is used in [15] chapter 3. i.e.

$$\hat{\mathbf{H}}_{tot} = \frac{\omega_a}{2}\hat{\boldsymbol{\sigma}}_z + \sum_k \omega_k \hat{\mathbf{b}}_k^\dagger \hat{\mathbf{b}}_k + \sum_k (g_k \hat{\mathbf{b}}_k + g_k \hat{\mathbf{b}}_k^\dagger)(\hat{\boldsymbol{\sigma}}_+ + \hat{\boldsymbol{\sigma}}_-) \tag{1.86}$$

For an introductory synopsis on spin algebras for the two-level system and/or the ladder operators of the quantum harmonic oscillator being used here, we recommend the introductory quantum textbook [6]. The respective GKSL equation of such a model may be computed to be

$$\partial_t \hat{\boldsymbol{\rho}}_{S_t} = \frac{-i}{2}(\omega_a + \Delta\omega_a)[\hat{\boldsymbol{\sigma}}_z, \hat{\boldsymbol{\rho}}_{S_t}] + \gamma D[\hat{\boldsymbol{\sigma}}_-]\hat{\boldsymbol{\rho}}_{S_t}. \tag{1.87}$$

where $D[\hat{\boldsymbol{\sigma}}_-](\hat{\boldsymbol{\rho}}) = \hat{\boldsymbol{\sigma}}_- \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\sigma}}_+ - \frac{1}{2}(\hat{\boldsymbol{\sigma}}_+ \hat{\boldsymbol{\sigma}}_- \hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\rho}} \hat{\boldsymbol{\sigma}}_+ \hat{\boldsymbol{\sigma}}_-)$, [15] for a derivation. $\hat{\boldsymbol{\sigma}}_z$ is an element of the Pauli matrices. The constants $\Delta\omega_a$ and γ depend on the environmental frequencies ω_k and the coupling parameters g_k. Letting $|0\rangle$, and $|1\rangle$ be some basis for $\mathbb{C}^2$, the Pauli matrices will have the following representation.

$$\hat{\boldsymbol{\sigma}}_x = |0\rangle\langle 1| + |1\rangle\langle 0| \tag{1.88}$$

$$\hat{\boldsymbol{\sigma}}_x = i|0\rangle\langle 1| - i|1\rangle\langle 0| \tag{1.89}$$

$$\hat{\boldsymbol{\sigma}}_z = |1\rangle\langle 1| - |0\rangle\langle 0| \tag{1.90}$$

$$\hat{\boldsymbol{\sigma}}_+ = 2(\hat{\boldsymbol{\sigma}}_x + i\hat{\boldsymbol{\sigma}}_y) \tag{1.91}$$

$$\hat{\boldsymbol{\sigma}}_- = 2(\hat{\boldsymbol{\sigma}}_x - i\hat{\boldsymbol{\sigma}}_y) \tag{1.92}$$

The operators $\hat{\mathbf{b}}$ and $\hat{\mathbf{b}}^\dagger$ are the ladder operators for the QSHO discussed in (1.24) and have the following properties for an arbitrary number state $|n\rangle$

$$\hat{\mathbf{b}}|n\rangle = \sqrt{n}|n-1\rangle,\ \ \hat{\mathbf{b}}|0\rangle = 0 \tag{1.93}$$

$$\hat{\mathbf{b}}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{1.94}$$

We now solve (1.87). The most general form a the density operator $\hat{\boldsymbol{\rho}}_{S_t} \in \mathcal{S}(\mathbb{C}^2)$ can take is

$$\hat{\boldsymbol{\rho}}_{S_t} = \frac{1}{2}[\mathbb{I}_S + x(t)\hat{\boldsymbol{\sigma}}_x + y(t)\hat{\boldsymbol{\sigma}}_y + z(t)\hat{\boldsymbol{\sigma}}_z] \tag{1.95}$$

Coherences (off-diagonal entries) are present because of the $\hat{\boldsymbol{\sigma}}_y$ and $\hat{\boldsymbol{\sigma}}_x$ terms. We will therefore be able to monitor decoherence by analyzing the functions $x(t)$ and $y(t)$. The scalar functions $x(t)$, $y(t)$, and $z(t)$ are computed in the following way.

- $\partial_t z(t) = Tr\{\hat{\boldsymbol{\sigma}}_z \partial_t \hat{\boldsymbol{\rho}}_{S_t}\}$
- $\partial_t y(t) = Tr\{\hat{\boldsymbol{\sigma}}_y \partial_t \hat{\boldsymbol{\rho}}_{S_t}\}$
- $\partial_t x(t) = Tr\{\hat{\boldsymbol{\sigma}}_x \partial_t \hat{\boldsymbol{\rho}}_{S_t}\}$

Using the GKSL equation to substitute for $\frac{\partial}{\partial t}\hat{\boldsymbol{\rho}}_{S_t}$ and $\hat{\boldsymbol{\rho}}_{S_t}$ using (1.95) and the above differential equations for $x(t)$, $y(t)$ and $z(t)$ we have

- $\partial_t z(t) = -\gamma(z(t) + 1)$
- $\partial_t y(t) = (\Delta\omega_a)x(t) - \frac{\gamma}{2}y(t)$
- $\partial_t x(t) = -(\Delta\omega_a)y(t) - \frac{\gamma}{2}x(t)$.

Assuming that the atom is initially in the excited state, i.e. $x(0) = 0$, $y(0) = 0$ and $z(0) = 1$, we get the following solution to (1.87).

- $z(t) = 2e^{-\gamma t} - 1$
- $y(t) = -e^{-\frac{\gamma t}{2}} \sin((\omega_a + \Delta\omega_a)t)$
- $x(t) = e^{-\frac{\gamma t}{2}} \sin((\omega_a + \Delta\omega_a)t)$.

The solution to the GKSL equation in the spontaneous emission case is therefore the following density operator.

$$\hat{\boldsymbol{\rho}}_{S_t} = e^{-\gamma t}\big|1\big\rangle\big\langle 1\big| + \frac{1+i}{2}e^{-\gamma t}\sin\left((\omega_a + \Delta\omega_a)t\right)\big|1\big\rangle\big\langle 0\big| + \tag{1.96}$$

$$\frac{1-i}{2}e^{-\gamma t}\sin\left((\omega_a + \Delta\omega_a)t\right)\big|0\big\rangle\big\langle 1\big| + \left(1 - e^{-\gamma t}\right)\big|0\big\rangle\big\langle 0\big| \tag{1.97}$$

Notice the exponential decay! This was to be expected! Also, notice the decoherence, i.e. $\mathcal{DK}(\hat{\boldsymbol{\rho}}_{S_t}) \to 0$ as $t \to \infty$ (using the decoherence measure defined in Definition 1.3.5). The off-diagonal entries decay as expected. In the limit $t \to \infty$ this state converges the ground state $\big|0\big\rangle\big\langle 0\big|$. A caricaturistic depiction of this setting is exhibited on the following page (Figure 1.1). Notice that had we ignored any interactions with the vacuum, the two-level atom would have evolved unitarily via the Hamiltonian $\hat{\mathbf{H}}_S = \frac{\omega_a \hat{\boldsymbol{\sigma}}_z}{2}$. However, due to the initial state being $\big|1\big\rangle\big\langle 1\big|$, there would be no dynamics because the excited state is an eigenvector of the Pauli matrix $\hat{\boldsymbol{\sigma}}_z$.

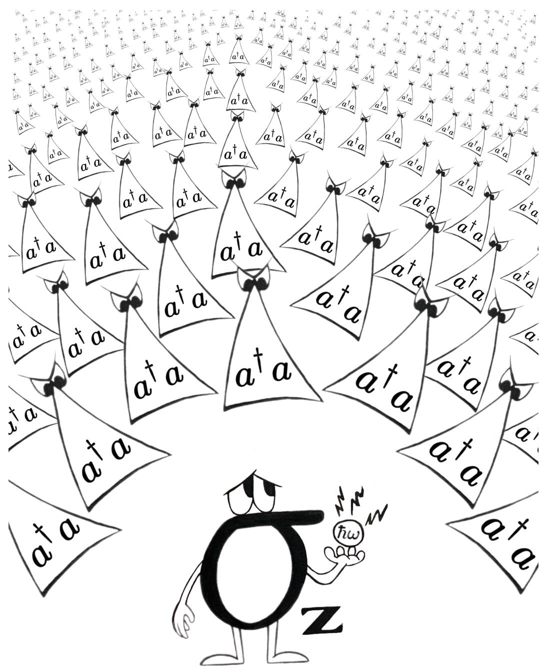

Figure 1.1: *A caricaturistic depiction of a single energized spin in a bath of oscillators. The vast bath of oscillators greedily yearning for the energy $\hbar\omega$ of the spin system. Greatly outnumbered and outmatched, the spin system has no choice but to give up its energy. Artwork by* Timothy Martinez @timbosculpt.

1.6.2 Collisional Decoherence

Let $|x\rangle$ represent the generalized eigenvector of the position operator $\hat{\mathbf{X}}$ of a mass point of some mesoscopic object (system) S, $|E\rangle$ will represent the state of a scattered particle or particles (environment) E. We will consider a simple case where the system S is large enough that it experiences no recoil from the scattered particles constituting the environmental degrees of freedom E. Some examples of systems with no recoil would be large molecules, dust particles, or even something like a bowling ball scattering photons (although describing a bowling ball with quantum mechanics is rather tenuous). The photons in all of these cases will have virtually no effect on the trajectory of the systems due to the size disparity between the systems in question and the scattering particles; a recoilless approximation is therefore appropriate. The recoilless scattering dynamics will now be summarized by the appropriate $S-matrix$ (scattering matrix) as it is done in [18] [12].

$$|x\rangle \otimes |E\rangle \xrightarrow{t} |x\rangle \otimes |E_x\rangle = |x\rangle \otimes \hat{\mathbf{S}}_x|E\rangle. \tag{1.98}$$

The $S-matrix$, $\hat{\mathbf{S}}_x$ is a unitary operator that maps some initial state of incoming particles, say photons, to the final state (scattered state). The nature of the $S-matrix$ will indeed depend on the type of particles being scattered off the center x, the type of particle doing the scattering, the forces involved, and the initial velocities; the details regarding generalities of $S-matrix$ theory lie beyond the scope of the present discussion so we will omit them referring the interested reader to the relevant discussion in [18] [12]. In [18] such a scattering setup is considered in order to compute decoherence time scales for mesoscopic systems. There, an arbitrary wave function of the total system S and E evolves as depicted below, for S in the initial states $\phi(x) \in L^2(\mathbb{R})$ and E in the initial state $|E\rangle \in \mathscr{H}_E$.

$$\Big(\int \phi(x)|x\rangle\Big)dx \otimes |E\rangle \xrightarrow{t} \int \phi(x)|x\rangle \otimes \hat{\mathbf{S}}_x|E\rangle dx, \tag{1.99}$$

the associated time-evolved density operator is therefore

$$\int\int \phi(x)\phi^*(y)|x\rangle\langle y| \otimes |E_x\rangle\langle E_y|dxdy. \tag{1.100}$$

The reduced density operator may be easily computed to be

$$Tr_E\Big\{\int\int \phi(x)\phi^*(y)|x\rangle\langle y| \otimes |E_x\rangle\langle E_y|dxdy\Big\} = \tag{1.101}$$

$$\int\int \phi(x)\phi^*(y)|x\rangle\langle y|Tr_E\Big\{|E_x\rangle\langle E_y|\Big\}dxdy = \int\int \phi(x)\phi^*(y)\langle E_y|E_x\rangle|x\rangle\langle y|dxdy. \tag{1.102}$$

The kernel $\langle E_y|E_x\rangle$ will yield non-unitary dynamics, the nature of which will depend on the properties of the particles being scattered; this includes the state of particles at $t=0$ amongst other things. If the initial state of E belongs to the subspace associated with the absolutely continuous spectrum of $\hat{\mathbf{S}}_x$, then in such a case the kernel $\langle E_y|E_x\rangle$ will only yield decoherence (See section 5.8). More generally, if the initial state of E belongs to the subspace associated with the *Rajchman* [58] spectrum of $\hat{\mathbf{S}}_x$ then once again we will only see non-unitary dynamics involving decoherence (See Section 5.8).

The latter details will be formalized in Chapter 5, for now, we will focus on the physically derived case where $\langle E_y | E_x \rangle$ is a function that becomes small as $|x-y|$ becomes large; this follows from the relevant discussions in [18] [12].

Photon scattering, Long-Wavelength limit

Let us fix x and x' for the moment. Assume that E are environmental photons and S a sphere so massive that it undergoes no recoil when scattering photons; if the wavelength of the incoming photons satisfies $\lambda >> |x-x'|$, then a single scattering event will not resolve the distance $|x-x'|$ i.e. coherences of distances on this order will not instantaneously disappear. They will decay exponentially. Using photons restricted to the above condition it can be shown [18](other zeh and joos paper) that

$$\langle E|\hat{\mathbf{S}}_{x'}^\dagger \hat{\mathbf{S}}_x|E\rangle \approx e^{-\Lambda t(x-x')^2} \tag{1.103}$$

for n photon scattering events, where the term Λt depends on n. i.e. the number of scattering events n is directly proportional to t. The relationship is the following, $t = \frac{n}{L^2 * flux}$, L is the length used to normalize the momentum wave functions [18]. The termΛ is the scattering constant, it represents the physical properties of the system-environment interaction. This constant is proportional to the size of the systems, i.e. a bowling ball will have a Λ that is significantly greater than the Λ of a dust particle. Details regarding the computation of Λ may be found in [18]. In the density operator representation, we can summarize the dynamics for the state of the systems $\phi(x) \in L^2(\mathbb{R})$ at $t=0$ as follows.

$$\hat{\boldsymbol{\rho}}_{S_0} := \int\int \phi(x)\phi^*(x')\big|x\big\rangle\big\langle x'\big|dxdy \xrightarrow{t} \hat{\boldsymbol{\rho}}_{S_t} := \int\int \phi(x)\phi^*(x')e^{-\Lambda t(x-x')^2}\big|x\big\rangle\big\langle x'\big|dxdx'. \tag{1.104}$$

In Figure 1.2 on the next page, we present a fun caricaturistic interpretation of this model.

Focusing on the kernels, the time-evolved kernel is.

$$K_S(x,x',t) := \phi(x)\phi^*(x')e^{-\Lambda t(x-x')^2} \tag{1.105}$$

A time derivative of the latter yields the differential equation

$$\partial_t K_S(x,x',t) = -\Lambda(x-x')^2 K_S(x,x',t). \tag{1.106}$$

Which is equivalent to the following density operator equation.

$$\partial_t \hat{\boldsymbol{\rho}}_{S_t} = -\Lambda\big[\hat{\mathbf{X}}, \big[\hat{\mathbf{X}}, \hat{\boldsymbol{\rho}}_{S_t}\big]\big] \tag{1.107}$$

This is indeed the nonunitary part of the master equation of the recoilless scattering model. As evidence, look at the kernel 1.105. It is clearl that the off-diagonal entries are decaying exponentially (i.e. $x \neq x'$). If the latter is not a satisfactory argument, one may compute the purity of $\hat{\boldsymbol{\rho}}_{S_t}$ to find that it is less than 1 for $t > 0$, unfortunately computing $\mathcal{DK}$ (Definition 1.3.5) is not an easy task in this case; it may nevertheless be done.

$$Tr\big\{\hat{\boldsymbol{\rho}}_{S_t}^2\big\} = \tag{1.108}$$

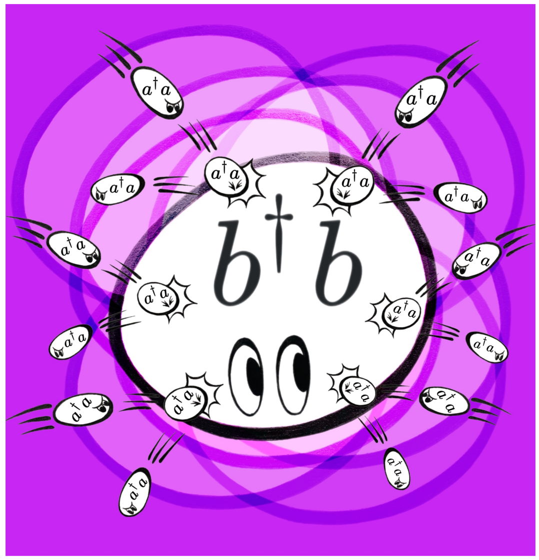

Figure 1.2: *A caricaturistic depiction of collisional decoherence. The superpositional nature of the large oscillator is suppressed by the interaction (via elastic scattering) with the smaller environmental oscillators. Artwork by Artwork by Timothy Martinez @timbosculpt.*

$$Tr\Big\{\int\int\int\int \phi(x)\phi^*(x')e^{-\Lambda t(x-x')^2}\phi(y)\phi^*(y')e^{-\Lambda t(y-y')^2}\big|x\big\rangle\big\langle x'\big|y\big\rangle\big\langle y'\big|dxdx'dydy'\Big\} = \tag{1.109}$$

$$Tr\Big\{\int\int\int \phi(x)\phi^*(x')e^{-\Lambda t(x-x')^2}\phi(x')\phi^*(y')e^{-\Lambda t(x'-y')^2}\big|x\big\rangle\big\langle y'\big|dxdx'dy'\Big\} = \tag{1.110}$$

$$\int\int |\phi(x)|^2|\phi^*(x')|^2e^{-2\Lambda t(x-x')^2}dxdx' = \int\int |K_S(x,x',0)|^2e^{-2\Lambda t(x-x')^2}dxdx' < 1 \tag{1.111}$$

where we have used the generalization of Theorem 1.3.1 discussed in [47] when going from (1.110) to (1.111). The exponential term damps the integrand in the latter. Hence, for large t the integrand approaches zero and therefore $\gamma(\hat{\boldsymbol{\rho}}_{S_t}) = Tr\big\{\hat{\boldsymbol{\rho}}^2_{S_t}\big\} \to 0$.

Since the scattering process does not affect the trajectory of our system particle we may include the intrinsic dynamics $\hat{\mathbf{H}}_S = \frac{1}{2m}\hat{\mathbf{P}}^2$ into the latter equation to render the full master equation

$$\hat{\boldsymbol{\rho}}_{S_t} = -i\big[\frac{1}{2m}\hat{\mathbf{P}}^2, \hat{\boldsymbol{\rho}}_{S_t}\big] - \Lambda\big[\hat{\mathbf{X}}, \big[\hat{\mathbf{X}}, \hat{\boldsymbol{\rho}}_{S_t}\big]\big]. \tag{1.112}$$

Notice that the position operator $\hat{\mathbf{X}}$ above is the only collapse operator. Values for Λ are given in Table 2.1 for two differently sized dust particles undergoing scattering interactions with varying environments. Table 2.1 has been taken from [18] chapter 3.

Prior to scrutinizing equation (1.112) further, let us refocus our attention back to the case where there is no self dynamics for the system $\hat{\mathbf{H}}_S$; this encapsulates the non-unitary dynamics pertaining to the recoilless processes. As expected, the recoilless aspect of this interaction means that there is no

Table 1.1: Λ in $cm^{-2}s^{-1}$ for two sizes of "dust particles" and various types of scattering processes evolving according to (1.112). This quantity measures how fast interference between different positions disappears for the long-wavelength limit. The figure is taken from [18] chapter 3 page 66. CBR below is an acronym for cosmic background radiation.

Environment	Λ **for dust grain,** $10^{-3}cm$	Λ **for dust particle,** $10^{-5}cm$
CBR	10^{6}	10^{-6}
300k photons	10^{19}	10^{12}
Sunlight on earth	10^{21}	10^{17}
Air molecules	10^{36}	10^{32}
Laboratory vacuum	10^{23}	10^{19}

dissipation; but as we have already mentioned, this does not mean that decoherence is not present. We have already elucidated on this comment by computing the purity of the evolved state in (1.108), however, to further illustrate the effects of decoherence let us take a generic state for some mesoscopic system to be in a simple superposition at $t = 0$ as is done in [18] chapter 3.

$$\phi(x) = N_1 e^{-(x-a_1)^2} + N_2 e^{-(x-a_2)^2} \tag{1.113}$$

Such a state characterizes a typical non-local state of matter. Assuming that our sphere S is in such a state, we can study the decay of the "off-diagonal" lumps of the associated density operator's kernel after scattering has taken place for an amount of time t. In Figure Figure 1.3 below a depiction of the time evolution of this Gaussian superposition (1.113) is provided. Figure 1.3 is taken from [18] chapter 3. Note the decaying in the off-diagonal entries $K_S(x, x', t)$, in particular those belonging to the off-diagonal lumps. A decoherence time scale may be defined as $\tau_{\Delta x} := \frac{1}{\Lambda(\Delta x)^2}$, $\Delta x = |x - x'|$, this encapsulates the rate at which decoherence takes place. Smaller decoherence time scales correspond to larger separations Δx. The larger the particle is, the larger Λ is, and therefore the smaller $\tau_{\Delta x}$ is for a respective Δx. Notice the differences in the values for Λ of dust particles of diameter $10^{-3}cm$ and 10^{-5}cm presented in Table 1.1. In contrast to Λ, it was shown in [31] that the τ_{Δ}, in units of seconds, pertaining to the case of the larger dust particle are at least four orders of magnitude larger than the corresponding values pertaining to the smaller dust particle. In Table 1.2 a single size for dust particles is selected, this time login in values of $\tau_{\Delta x}$ for a fixed value of Δx. Notice how fast these coherences dissipate, even a vacuum would decohere any positional coherences of sizes comparable to the size of our particle S in 10^{-14} seconds, and these are decoherence time scales for a dust particle of size $10^{-3}cm$. Decoherence time scales for objects much larger, or much more classical, would be many orders of magnitude smaller. This is consistent with our day-to-day classical world experiences, in which we never perceive any macroscopic object or mesoscopic object to be in a superposition; although the superpositions exist in theory, their lifespans are too short for us to perceive them with the naked eye.

Let us now return to equation (1.112) which more generally describes the dynamics of a free particle undergoing scattering interactions with some bosonic environments. As we already showed, the non-unitary term will dissipate the off-diagonal terms of the density operator $\hat{\boldsymbol{\rho}}_{S_t}$ as time progresses; the

Table 1.2: $\tau_{\Delta x}$ is seconds for a large dust particle, Δx is equal to the diameter of the particle, and various types of scattering processes. This table is taken from [12] chapter 3 page 135.

Environment	$\tau_{\Delta x}$ **for Dust grain,** $10^{-3}cm$
Cosmic background radiation	1
Photons at room temperature	10^{-18}
Best laboratory vacuum	10^{-14}
Air at normal pressure	10^{-31}

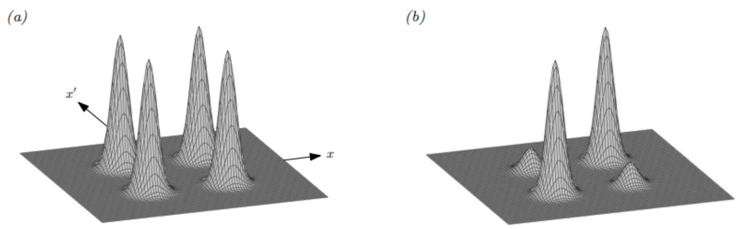

Figure 1.3: *From Schlosshauer's paper "Quantum Decoherence" 2019. Collisional decoherence of a density matrix representing a Gaussian wave packet as generated by equation (1.107) with the initial state (1.113). The images a) and b) represent $|K_S(x, x', t)|^2$ before and after decoherence respectively.*

unitary evolution term $-i\left[\frac{1}{2m}\hat{\mathbf{P}}^2, \hat{\boldsymbol{\rho}}_{S_t}\right]$ from (1.112) will then be responsible for the free spreading of the wave packet. It can be shown that the free particle Hamiltonian has the effect of spreading localized wave packets in x ([6] chapter 6) into highly unlocalized ones. As seen in figure 4, a symmetric Gaussian initial state will become flat in the off-diagonal entries $x = -x'$ and will extend along the diagonal $x = x'$ which represents the probability distribution of our particles position $P(x, t) := K_S(x, x, t)$.

Short-wavelength limit

If the wavelength of the scattered environmental particles is much smaller than a coherent separation $\Delta x = |x - x^{'}|$, then these environmental particles can resolve such a separation in a single scattering event. This in turn leads to maximum spatial decoherence per scattering event [12] [18]. In this limit, the decay of the coherence terms of the density matrix will not depend on $|x - x'|$, instead we have

$$K_S(x, x', t) = K_S(x, x', 0)e^{-\Gamma_{tot}t} \tag{1.114}$$

where Γ_{tot} is a global decoherence rate. (1.114) should not come to us as a surprise since we have already seen it when dealing with spontaneous emission.

For both, the long-wavelength and short-wavelength limits of scattering decoherence there is one conspicuous property of the time-evolved states. This is the narrowing that takes place along the diagonal ($x = x'$) of the respective kernel. Were the off-diagonal entries to truly vanish completely this would imply that full decoherence with respect to the position basis had taken place. However, as the collisional model of decoherence exemplifies, energy is needed to induce decoherence; te energy in this case is expended by the colliding photons. Recalling that t was directly proportional to the number of photons scattered one may deduce that it would take infinite energy to induce full decoherence (i.e.an infinite amount of time). Furthermore, excluding the self-dynamics of the system S and the photonic field E leads us to overlook any coherences that might ensue dynamically from the latter. If one were to take a much more descriptive model, one that included the self-dynamics of both S and E, then one would not see the indefinite narrowing along the diagonal of $K_S(x, x', t)$ as t became arbitrarily large; in lieu of this one would see a steady state solution to the master equation arise, one with a limiting narrowing along $(x = x')$ which is referred to as the *limiting coherence length*. A study of such models may be found in [32][33]. For many mesoscopic physical systems which scatter light, the decoherence time scales are so small that one may simply introduce a cut-off time T comparable to the decoherence time pertaining to any resolvable coherences; within such a time domain the decoherence dynamics will be approximately faithful to the actual dynamics so long as the self-dynamics of S and E are much slower than the dynamics induced by decoherence. In order to further motivate the collisional decoherence model we will present the *Quantum Brownian Motion* [31] in what follows. We will not be solving this model nor discussing its limiting decoherence length here.

Quantum Brownian Motion

Possibly the most celebrated decoherence model is *Quantum Brownian Motion* (QBM). This is a model describing the dynamics of a particle weakly coupled to a thermal bath of non-interacting harmonic oscillators. The self-Hamiltonian of the environment being a linear combination of QSHOs (1.23)

$$H_E = \sum_i \left(\frac{1}{2m_i}\hat{\mathbf{P}}_i^2 + \frac{1}{2} m_i \omega_i^2 \hat{\mathbf{Q}}_i^2\right), \tag{1.115}$$

where m_i and ω_i are the mass and natural frequency of the ith oscillator while $\hat{\mathbf{Q}}_i$ and $\hat{\mathbf{P}}_i$ denote the canonical positions and momenta operators. The interaction Hamiltonian is taken to be

$$H_I = \hat{\mathbf{X}} \otimes \sum_i c_i \hat{\mathbf{Q}}_i, \tag{1.116}$$

a bilinear coupling of the system's position $\hat{\mathbf{X}}$ to the positions $\hat{\mathbf{Q}}_i$ of the environmental oscillators. Finally, it is assumed that the system (the particle) S will have oscillatory self-dynamics. i.e

$$H_S = \frac{1}{2M}\hat{\mathbf{P}}^2 + \frac{1}{2} M \Omega^2 \hat{\mathbf{X}}^2, \tag{1.117}$$

where M is the mass of the particle and Ω is its natural frequency. Using the *Born* and *Markov* approximations as well as some approximations analogous to what was done for the case of spontaneous emission (see [12] chapter 3 and [18] chapter 3)it can be shown that the corresponding Master equation is the following.

$$\partial_t \hat{\boldsymbol{\rho}}_{S_t} = -i\Big[\hat{\mathbf{H}}_S + \frac{1}{2} M \Delta^2 \hat{\mathbf{X}}^2,\, \hat{\boldsymbol{\rho}}_{S_t}\Big] - i\gamma\Big[\hat{\mathbf{X}},\, \{\hat{\mathbf{P}},\, \hat{\boldsymbol{\rho}}_{S_t}\}\Big] - D\Big[\hat{\mathbf{X}},\, [\hat{\mathbf{X}},\, \hat{\boldsymbol{\rho}}_{S_t}]\Big] - f\Big[\hat{\mathbf{X}},\, [\hat{\mathbf{P}},\, \hat{\boldsymbol{\rho}}_{S_t}]\Big]. \tag{1.118}$$

Following [31], we present below a list defining all of the constants present in such a master equation where $J(\omega)$ is the spectral density of the environment.

- $\nu(\tau) = \int_0^\infty d\omega J(\omega)\coth(\frac{\hbar\omega}{2k_B T})cos(\omega\tau)$, noise-kernel.
- $\eta(\tau) = \int_0^\infty d\omega J(\omega)\sin(\omega\tau)$, dissipation kernel.
- $\Delta^2 = -\frac{2}{M}\int_0^\infty d\tau\eta(\tau)\cos(\Omega\tau)$, the square of the shifted natural frequency of the particle.
- $\gamma = \frac{2}{M\Omega}\int_0^\infty d\tau\eta(\tau)\sin(\Omega\tau)$, damping rate due to dissipation effects.
- $D = \frac{1}{\hbar}\int_0^\infty d\tau\nu(\tau)\cos(\Omega\tau)$, scattering constant analogous to Λ in the previous section.
- $f = -\frac{1}{M\Omega}\int_0^\infty d\tau\nu(\tau)\sin(\Omega\tau)$, also represents decoherence but usually negligible, especially at high temperatures.

It can be shown that dispersion in position may be given by $(\Delta\hat{\mathbf{X}})^2(t) = \frac{D}{2m^2\gamma^2}t$ [12] [31] [18]. The ensemble width $\Delta\hat{\mathbf{X}}(t)$ (the variance of the operator $\hat{\mathbf{X}}$ with respect to the state $\hat{\boldsymbol{\rho}}_{S_t}$) therefore scales asymptotically as $\sqrt{t}$ which is the scaling behavior seen in classical Brownian motion, hence the name QBM [31]. Notice that in the regime where dissipation affects, γ, and low-temperature effects, f, may be neglected (1.118) is approximately the collisional decoherence master equation seen in (1.112), but now with a QSHO self-Hamiltonian for the system S. However, if the mass of the S is large enough, then $\hat{\mathbf{H}}_S$ may be approximated by a free-particle Hamiltonian. We, therefore, see that collisional decoherence (1.107) is just a special case of the QBM model (1.118). It is in this sense that the interaction von Neumann (quantum measurement regime [31]) interaction Hamiltonian $H_I = \hat{\mathbf{X}} \otimes \sum_i c_i\hat{\mathbf{Q}}_i$, which will be the regime of focus for the main work in this thesis, is seen as physically viable. i.e. in the sense that $\hat{\mathbf{H}}_I \approx \hat{\mathbf{H}}_{tot}$, where $\hat{\mathbf{H}}_{tot}$ is the total QBM Hamiltonian for appropriate time domains.

1.6.3 Multipartite Open Quantum Systems

Let $\mathscr{H}_S \otimes \bigotimes_{k=1}^N \mathscr{H}_{E^k}$ be some arbitrary tensor product *Hilbert* space. Let us consider a separable density operator as the initial state of some multipartite quantum system evolving in $\mathcal{S}\big(\mathscr{H}_S \otimes \bigotimes_{k=1}^N \mathscr{H}_{E^k}\big)$, namely

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=1}^{N} \hat{\boldsymbol{\rho}}^{E_0^k} \in \mathcal{S}\big(\mathscr{H}_S \otimes \bigotimes_{k=1}^{N} \mathscr{H}_{E^k}\big). \tag{1.119}$$

Here we consider a quantum system S interacting with N macroscopic environments E^k; we write the subscript 0 in E_0^k in order to emphasize that this is the initial state of the kth environment E^k, similarly, we use the subscript S_0 to highlight the initial state of the system S. We will assume that the time evolution of (1.119) lies within the *quantum-measurement limit* regime [31], i.e. $\hat{\mathbf{H}}_{tot} \approx \hat{\mathbf{H}}_{int}$. Let us assume a *von Neumann* type interaction Hamiltonian (this definition is taken from [18]). i.e.

$$\hat{\mathbf{H}}_{int} = \hat{\mathbf{X}} \otimes \sum_{k=1}^{N} g_k \hat{\mathbf{B}}_k \tag{1.120}$$

where the operators $\hat{\mathbf{X}}$ and $\hat{\mathbf{B}}_k$ above are respectively the position operator and some arbitrary observable; each acting on its respective Hilbert space, i.e. all of the $\hat{\mathbf{B}}_k$ act on different Hilbert spaces. The constants g_k are coupling strengths between the position operator $\hat{\mathbf{X}}$ of S and the observable $\hat{\mathbf{B}}_k$ of the kth environment E^k.The corresponding time evolution operator is therefore

$$\hat{\mathbf{U}}_t = e^{-it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N} g_k\hat{\mathbf{B}}_k}. \tag{1.121}$$

We evolve our total initial state using the evolution operator (1.121).

$$\hat{\boldsymbol{\rho}}_t = \left(e^{-it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N} g_k\hat{\mathbf{B}}_k}\right)\hat{\boldsymbol{\rho}}_{S_0}\otimes\bigotimes_{k=1}^{N}\hat{\boldsymbol{\rho}}^{E_0^k}\left(e^{it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N} g_k\hat{\mathbf{B}}_k}\right). \tag{1.122}$$

We will now do something which is divergent from the methods applied up until now for the study of quantum open systems. Rather than trace out all of the environmental degrees of freedom, we shall be tracing out only a subset of these. i.e. we shall be studying the state of the subsystem formed by the system S and the first N_E environments. We shall take the partial trace of the time-evolved density operator (5.4) over the remaining $M_E := N - N_E$ environments. We present this partial trace as a lemma.

Lemma 1.6.1 (Multiple Partial Traces)

$$Tr_{E^{N_E+1},E^{N_E+2},\ldots,E^N}\left\{\hat{\boldsymbol{\rho}}_t\right\} = \mathscr{U}_{N_E,t}\left(\mathscr{E}_t\left(\hat{\boldsymbol{\rho}}_s\right)\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k}\right). \tag{1.123}$$

Where

$$\mathscr{U}_{n,t}\left(\hat{\mathbf{A}}\right) := e^{-it\hat{\mathbf{X}}\otimes\hat{\mathbf{S}}_n}\left(\hat{\mathbf{A}}\right)e^{it\hat{\mathbf{X}}\otimes\hat{\mathbf{S}}_n} \tag{1.124}$$

$$\hat{\mathbf{S}}_n := \sum_{k=1}^{n} g_k\hat{\mathbf{B}}_k \tag{1.125}$$

and

$$\mathscr{E}_t^{M_E}\{\hat{\boldsymbol{\sigma}}\} := \int\int\langle x|\hat{\boldsymbol{\sigma}}|y\rangle\Gamma_{M_E}(t,x,y)|x\rangle\langle y|dxdy. \tag{1.126}$$

where

$$\Gamma_{M_E}(t,x,y) := \prod_{k=N_E+1}^{N} Tr_k\left\{e^{-itxg_k\hat{\mathbf{B}}_k}\hat{\boldsymbol{\rho}}^{E_0^k}e^{ityg_k\hat{\mathbf{B}}_k}\right\} \tag{1.127}$$

$M_E = N - N_E$, the number of traces being taken in equation (1.127).

Proof.

$$Tr_{E_{N_E+1},E_{N_E+2},\ldots,E_N}\left\{\hat{\boldsymbol{\rho}}_t\right\} = \tag{1.128}$$

$$Tr_{E_{N_E+1},E_{N_E+2},\ldots,E_N}\left\{e^{-it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N} g_k\hat{\mathbf{B}}_k}\left(\hat{\boldsymbol{\rho}}_{S_0}\otimes\bigotimes_{k=1}^{N}\hat{\boldsymbol{\rho}}^{E_0^k}\right)e^{it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N} g_k\hat{\mathbf{B}}_k}\right\} = \tag{1.129}$$

$$\mathscr{U}_{N_E,t}\left(Tr_{E_{N_E+1},E_{N_E+2},\ldots,E_N}\left\{e^{-it\hat{\mathbf{X}}\otimes\sum_{k=N_E+1}^{N} g_k\hat{\mathbf{B}}_k}\left(\hat{\boldsymbol{\rho}}_{S_0}\otimes\bigotimes_{k=N_E+1}^{N}\hat{\boldsymbol{\rho}}^{E_0^k}\right)e^{it\hat{\mathbf{X}}\otimes\sum_{k=N_E+1}^{N} g_k\hat{\mathbf{B}}_k}\right\}\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k}\right) \tag{1.130}$$

Let us now use the generalized eigenvectors of $\hat{\mathbf{X}}$ in order to write $\hat{\boldsymbol{\rho}}_S = \int\int K_S(x,y)|x\rangle\langle y|dxdy$ where $K_S(x,y) = \langle x|\hat{\rho}_S|y\rangle$. Using the latter,

$$e^{-it\hat{\mathbf{X}}\otimes\sum_{k=N_E+1}^{N} g_k\hat{\mathbf{B}}_k}\left(\hat{\boldsymbol{\rho}}_{S_0}\otimes\bigotimes_{k=N_E+1}^{N}\hat{\boldsymbol{\rho}}^{E_0^k}\right)e^{it\hat{\mathbf{X}}\otimes\sum_{k=N_E+1}^{N} g_k\hat{\mathbf{B}}_k} = \tag{1.131}$$

$$\int\int K_S(x,y)|x\rangle\langle y|\left(e^{-itx\sum_{k=N_E+1}^{N} g_k\hat{\mathbf{B}}_k}\left(\bigotimes_{k=N_E+1}^{N}\hat{\boldsymbol{\rho}}^{E_0^k}\right)e^{ity\sum_{k=N_E+1}^{N} g_k\hat{\mathbf{B}}_k}\right)dxdy = \tag{1.132}$$

$$\int\int K_S(x,y)|x\rangle\langle y|\otimes\bigotimes_{k=N_E+1}^{N} e^{-itxg_k\hat{\mathbf{B}}_k}\hat{\boldsymbol{\rho}}^{E_0^k}e^{ityg_k\hat{\mathbf{B}}_k}dxdy. \tag{1.133}$$

Furthermore

$$Tr_{E_{N_E+1},E_{N_E+2},\ldots,E_N}\left\{\int\int K_S(x,y)|x\rangle\langle y|\otimes\bigotimes_{k=N_E+1}^{N} e^{-itxg_k\hat{\mathbf{B}}_k}\hat{\boldsymbol{\rho}}^{E_0^k}e^{ityg_k\hat{\mathbf{B}}_k}dxdy\right\} = \tag{1.134}$$

$$\int\int K_S(x,y)|x\rangle\langle y|Tr_{E_{N_E+1},E_{N_E+2},\ldots,E_N}\left\{\bigotimes_{k=N_E+1}^{N} e^{-itxg_k\hat{\mathbf{B}}_k}\hat{\boldsymbol{\rho}}^{E_0^k}e^{ityg_k\hat{\mathbf{B}}_k}\right\}dxdy = \tag{1.135}$$

$$\int\int K_S(x,y)\Gamma_{M_E}(t,x,y)|x\rangle\langle y|dxdy = \mathscr{E}_t^{M_E}\left(\hat{\boldsymbol{\rho}}_{S_0}\right) \tag{1.136}$$

Finally, using (1.130) and (1.136), we have

$$(1.130) = \mathscr{U}_{N_E,t}\left(\mathscr{E}_t^{M_E}\left(\hat{\boldsymbol{\rho}}_{S_0}\right)\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k}\right) \tag{1.137}$$

□

The density operators (1.137) describe a simple case of a multipartite open quantum system. We call them simple because there are no interactions between the environmental degrees of freedom. Such states are useful for the description of multiple observer monitoring states; these are states where each of the environmental degrees of freedom in (1.137) represent some physical system that the kth observer conducts measurements on in order to indirectly learn about the system S [28]. The dynamics of states of the form (1.137) will be the primary focus of this thesis; in particular, we shall be interested in answering the question of whether or not such a state converges to a so-called SBS state within some time domain of interest (See Chapters 4 and 5).

Chapter 2

Quantum Maps, Distance Measures, and Inequalities

This chapter is dedicated to introducing and motivating many of the tools that will be needed in proving the main results of this thesis. The three topics to be discussed here are *quantum maps*, *distance measures*, and relevant inequalities. Quantum maps play a key role in open quantum systems; these are maps that take density operators as inputs and return density operators as outputs whilst accounting for effects such as dissipation, decoherence, unitary evolution, and even a combination of all these. In close quantum systems, one encounters such maps when solving the LV equation. The solutions to this equation are a density operator which has been unitarily evolved from some initial state (1.30); such evolution is a basic example of a quantum map. In open quantum systems the notion of a quantum map is more nuanced due to its various applications. We will begin this chapter by formally defining the notion of a quantum map and discussing physical motivations. We will then continue with a discussion on various norm and metric inequalities that we shall be needing to study proximity between density operators being evolved by different quantum maps respectively; this will play a key role in the rest of this thesis.

2.1 Quantum Maps

Let us right away define a quantum map. We follow closely the definition presented in [9] (where the terminology *quantum operation* is used in lieu of quantum map.).

Definition 2.1.1 (Quantum Map)

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two arbitrary Hilbert spaces. We define a quantum map $\mathscr{E}$ as a map from the set of density operators of the input space $\mathcal{S}(\mathscr{H}_1)$ to the set of density operators for the output space $\mathcal{S}(\mathscr{H}_2)$, with the following three axiomatic properties.

- A1: $Tr\{\mathscr{E}(\hat{\boldsymbol{\rho}})\}$ is the probability that the process $\mathscr{E}$ occurs, when the $\hat{\boldsymbol{\rho}}$ is in the initial state. Thus, $0 \leq Tr\{\mathscr{E}(\hat{\boldsymbol{\rho}})\} \leq 1$ for any state $\hat{\boldsymbol{\rho}}$.
- A2: $\mathscr{E}$ is a *convex-linear map* on the set of density operators, i.e. for a probability distribution $\{p_i\}$,
$$\mathscr{E}\Big(\sum_i p_i \hat{\boldsymbol{\rho}}_i\Big) = \sum_i p_i \mathscr{E}(\hat{\boldsymbol{\rho}}_i) \tag{2.1}$$
- A3: $\mathscr{E}$ is a *completely positive* map. i.e., if $\mathscr{E}$ maps density operators of $\mathcal{S}(\mathscr{H}_1)$ to density operators of $\mathcal{S}(\mathscr{H}_2)$, then $\mathscr{E}(\hat{\mathbf{A}})$ must be positive for any positive operator $\hat{\mathbf{A}}$. Furthermore, let $\mathscr{H}_3$ a third arbitrary Hilbert space. It must then be true that $(\mathcal{I}\otimes\mathscr{E})(\hat{\mathbf{A}})$ is positive for any positive operator $\hat{\mathbf{A}} \in \mathcal{B}(\mathscr{H}_1 \otimes \mathscr{H}_3)$ where $\mathcal{I}$ is the identity map on $\mathcal{B}(\mathscr{H}_1)$.

In mundane terms, quantum maps are maps that respect the rules of quantum mechanics. Perhaps with the exception of the second half of $A3$, all of the properties in Definition 2.1.1 are quite natural. To understand the completely positive criteria let us first convince ourselves that complete positivity is more restrictive than positivity. Let $\hat{\boldsymbol{\rho}} \in \mathcal{S}(\mathbb{C}^2)$, using the basis $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ we have the following representation
$$\hat{\boldsymbol{\rho}} = a|1\rangle\langle 1| + b|0\rangle\langle 0| + c|1\rangle\langle 0| + c^*|0\rangle\langle 1|. \tag{2.2}$$
From basic linear algebra, we know that a matrix and its transpose have the same eigenvalues, $\mathscr{T}(\hat{\boldsymbol{\rho}}) := \hat{\boldsymbol{\rho}}^T$ is therefore also a positive operator. The transpose of a matrix is hence a positive map. Now, consider the density operator
$$\hat{\boldsymbol{\sigma}} := \frac{1}{2}\Big(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\Big) \;\; \in \mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2) \tag{2.3}$$
where $|00\rangle$ is shorthand for $|0\rangle \otimes |0\rangle$. Acting on (2.3) with the map $\mathscr{T} \otimes \mathcal{I}$, which executes a transpose in the subspace pertaining to the left-hand side of the tensor products $|0\rangle \otimes |0\rangle$ ... etc, while $\mathcal{I}$ is the identity map of the complementary subspace,
$$(\mathscr{T} \otimes \mathcal{I})(\hat{\boldsymbol{\sigma}}) = \frac{1}{2}\Big(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|\Big). \tag{2.4}$$
This matrix is known to have eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$, which means that it is not a positive matrix. We, therefore, see that complete positivity is a stronger condition than positivity. We have yet to motivate why complete positivity is compulsory. The physical motivation is this. In quantum open systems, dynamics will be generated by quantum maps. Consider some time-dependent trace-preserving quantum map $\mathscr{E}_t$, i.e. this map will be a bonafide trace-preserving quantum map for all

$t > 0$. Let this map evolve a quantum system S_1. One can embed such an open quantum system into a larger quantum open system, now including another system S_2 which is static in time and does not interact with S_1. The total dynamics of this larger quantum open system will now be described by $\mathscr{E}_t \otimes \mathcal{I}$ where $\mathcal{I}$ is the identity quantum map of the subsystem S_2. The map $\mathscr{E}_t \otimes \mathcal{I}$ will of course be expected to map the density operator to another density operator, it will be a positive map, hence the necessity for complete positivity.

In the previous chapter, we alluded to the fact that environmentally induced non-unitary dynamics (1.69) characterized by a family of identity-resolving operators (1.76), called Kraus operators, constitutes a quantum map. We have already seen examples of such maps and have seen that they respect the rules of quantum mechanics. Nevertheless, it would be a worthwhile exercise to prove that the axioms $A1 - A3$ of Definition 2.1.1 are satisfied by maps generated by a family of Kraus operators (1.75).

Let $\mathscr{H}$ be some arbitrary Hilbert space and let $\hat{\boldsymbol{\rho}} \in \mathcal{S}\big(\mathscr{H}_1\big)$. Now, let $\big\{\hat{\mathbf{M}}_i\big\}_i \in \mathcal{B}\big(\mathscr{H}_1\big)$ be a family of Kraus operators, i.e. $\sum_i \hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i = \mathbb{I}_1$, then the map $\mathscr{E}\big(\hat{\boldsymbol{\rho}}\big) := \sum_i \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger$ satisfies $A1 - A3$ of Definition 2.1.1. $A1$ is trivial to verify; the key element in the proof is the cyclicity of the trace (1.33).

$$Tr\{\mathscr{E}\big(\hat{\boldsymbol{\rho}}\big)\} = Tr\big\{\sum_i \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger\big\} = \sum_i Tr\big\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger\big\} = \sum_i Tr\big\{\hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}\ \big\} = \tag{2.5}$$

$$Tr\big\{\sum_i \hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}\ \big\} = Tr\big\{\hat{\boldsymbol{\rho}}\ \big\} = 1. \tag{2.6}$$

Proving $A2$ is also simple. Let $\big\{p_i, \hat{\boldsymbol{\rho}}_i\big\}_i$ be an ensemble of density operators, then

$$\mathscr{E}\Big(\sum_i p_i \hat{\boldsymbol{\rho}}_i\Big) = \sum_j \hat{\mathbf{M}}_j \Big(\sum_i p_i \hat{\boldsymbol{\rho}}_i\Big) \hat{\mathbf{M}}_j^\dagger = \sum_i p_i \sum_j \hat{\mathbf{M}}_j \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_j^\dagger = \sum_i p_i \mathscr{E}\big(\hat{\boldsymbol{\rho}}_i\big). \tag{2.7}$$

Finally, and most importantly, we will prove $A3$. We will first prove positivity. Let $\big|\psi\big\rangle \in \mathscr{H}_1$,

$$\big\langle\psi\big| \sum_i \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger \big|\psi\big\rangle = \sum_i \big\langle\psi\big|\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger\big|\psi\big\rangle = \sum_i \big\langle\phi_i\big|\hat{\boldsymbol{\rho}}\big|\phi_i\big\rangle \geq 0 \tag{2.8}$$

where $\big|\phi_i\big\rangle$ is the image of $\hat{\mathbf{M}}$ acting on $\big|\psi\big\rangle$. Owing to the positivity of $\hat{\boldsymbol{\rho}}$, all of the elements of the sum $\sum_i \big\langle\phi_i\big|\hat{\boldsymbol{\rho}}\big|\phi_i\big\rangle$ will be positive, leading us to conclude that $\mathscr{E}\big(\boldsymbol{\rho}\big)$ is positive (2.8). To show complete positivity we first let $\mathscr{H}_3$ be an arbitrary Hilbert space and let $\big|\eta\big\rangle \in \mathcal{B}\big(\mathscr{H}_1 \otimes \mathscr{H}_3\big)$. Furthermore, consider an arbitrary positive operator $\hat{\mathbf{A}} \in \mathcal{B}\big(\mathscr{H}_1 \otimes \mathscr{H}_3\big)$. Then,

$$\big\langle\eta\big|\big(\mathcal{I} \otimes \mathscr{E}\big)\big(\hat{\mathbf{A}}\big)\big|\eta\big\rangle = \big\langle\eta\big| \sum_i \big(\mathbb{I}_1 \otimes \hat{\mathbf{M}}_i\big) \hat{\mathbf{A}} \big(\mathbb{I}_1 \otimes \hat{\mathbf{M}}_i^\dagger\big)\big|\eta\big\rangle = \tag{2.9}$$

$$\sum_i \big\langle\eta\big|\big(\mathbb{I}_1 \otimes \hat{\mathbf{M}}_i\big) \hat{\mathbf{A}} \big(\mathbb{I}_1 \otimes \hat{\mathbf{M}}_i^\dagger\big)\big|\eta\big\rangle \tag{2.10}$$

Letting $\big|\chi_i\big\rangle := \big(\mathbb{I}_1 \otimes \hat{\mathbf{M}}_i\big)\big|\eta\big\rangle$ (a ket in $\mathscr{H}_1 \otimes \mathscr{H}_3$) we see that (2.10) is just

$$\sum_i \big\langle\chi_i\big|\hat{\mathbf{A}}\big|\chi_i\big\rangle \geq 0 \tag{2.11}$$

which is obviously a quantity greater than or equal to one owing to the positivity of $\hat{\mathbf{A}}$. Since the family of identity-resolving operators $\hat{\mathbf{M}}_i$ were taken to be general, we have actually proven that all trace-preserving (also known as *quantum channels* [9]), identity-resolving families of operators $\{\hat{\mathbf{M}}_i\}_i$ generate a quantum map. We have therefore proven one direction of the following theorem.

Theorem 2.1.1 (Quantum Channel Representation)

A trace-preserving map $\mathscr{E}$ is a *quantum channel* (i.e. trace-preserving) if and only if

$$\mathscr{E}(\hat{\boldsymbol{\rho}}) = \sum_i \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger, \tag{2.12}$$

for some set of operators $\{\hat{\mathbf{M}}_i\}_i$ which map the input Hilbert to the output Hilbert space, and $\sum_i \hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i = \mathbb{I}$

For the cases where we consider maps $\mathscr{E}$ which do not preserve the trace, Theorem 2.1.1 may be adapted to the following.

Theorem 2.1.2 (Quantum Map Representation)

A map $\mathscr{E}$ is a quantum map if and only if

$$\mathscr{E}(\hat{\boldsymbol{\rho}}) = \sum_i \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger, \tag{2.13}$$

for some set of operators $\{\hat{\mathbf{M}}_i\}_i$ which map the input Hilbert space to the output *Hilbert* space, and $0 \leq \sum_i \hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i \leq \mathbb{I}$

Note that the case where there is only one operator $\hat{\mathbf{M}}_i$ pertains to the unitary map case. i.e. if there is only one $\hat{\mathbf{M}}_i$, namely $\hat{\mathbf{M}}$, and we require that $\hat{\mathbf{M}}^\dagger \hat{\mathbf{M}} = \mathbb{I}$, then forcibly $\hat{\mathbf{M}}$ will be a unitary operator. In section 1.5 we saw that open systems evolve non-unitarily via the quantum map generated by the Kraus operators obtained from the partial tracing over the environmental degrees of freedom E. The Kraus operators were shown to have the identity resolution property (1.76); therefore preserving positivity and the unit trace property. More generally, the family $\{\mathbf{M}_i\}_i$ does not need to be countable. i.e. we could have a quantum map defined as $\mathscr{F}(\hat{\boldsymbol{\rho}}) := \int \hat{\mathbf{M}}_x \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_x^\dagger dx$. Such a map may be easily shown to preserve the unit trace and positivity properties.

GKSL Generator

There are cases where a time-dependent quantum map $\mathscr{E}_t$, such as the ones generating non-unitary evolution discussed in the previous section, will satisfy the following *semigroup* properties.

- $\mathscr{E}_t$ is strongly continuous.
- $\mathscr{E}_t \mathscr{E}_s = \mathscr{E}_{t+s}$, for al $t, s \geq 0$

When the above properties are satisfied, one may invoke the celebrated *Hille-Yosida* theorem [44] which states that there exists a densely defined generator $\mathcal{L}$ defined as

$$\mathcal{L}\hat{\boldsymbol{\rho}} := \lim_{t\to 0} \frac{1}{t}\big(\mathscr{E}_t(\hat{\boldsymbol{\rho}}) - \hat{\boldsymbol{\rho}}\big) \tag{2.14}$$

such that

$$\partial_t \mathscr{E}_t = \mathcal{L}\mathscr{E}_t \tag{2.15}$$

together with the initial condition $\lim_{t\to 0} \mathscr{E}_t \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\rho}}$. For unitary dynamics the generators $\mathcal{L}$ will just be those generated by some Hamiltonian. However, in the non-unitary case $\mathcal{L}$ may be the GKSL super operator that we have already seen in (1.85). i.e.

$$\mathcal{L}(...) := -i\big[\hat{\mathbf{H}}_S, ...\big] + \sum_i \gamma_i \Big(\hat{\mathbf{L}}_i(...)\hat{\mathbf{L}}_i^\dagger - \frac{1}{2}\Big\{\hat{\mathbf{L}}_i^\dagger \hat{\mathbf{L}}_i, ...\Big\}\Big). \tag{2.16}$$

We now see the tightly knitted connection between the GKSL equation and quantum maps, and may further appreciate the significant role that quantum maps play when modeling general quantum dynamics.

2.2 Quantum Map Examples

We will now present various examples of quantum maps, some of which we have seen already, highlighting their quantum map properties, and others which we will see in what is to come.

2.2.1 Phase Kick Decoherence

Let $\mathscr{H}_S = \mathbb{C}^2$, and consider the an arbitrary state $|\psi\rangle \in \mathbb{C}^2$. i.e. $|\psi\rangle = a|0\rangle + b|1\rangle$ where $|0\rangle$ and $|1\rangle$ are the eigenvetcors of the Pauli matrix $\hat{\boldsymbol{\sigma}}_z$. Such a state may experience an environmental kick emanating from environmental particles which interact deterministically with S. Such a kick may be generated by the unitary operator $\hat{\mathbf{R}}_z(\theta) := e^{-\frac{i\theta}{2}\hat{\boldsymbol{\sigma}}_z}$; in this case the kick is a rotation along the z-axis of the Bloch-Sphere by θ degrees [9] [15]. Assuming that some experimentalist is taking a measurement of S, subsequently after the phase kick, has no knowledge of the state of the environment interacting with S. Furthermore, assume that the statistics of the parameter θ are $\mu_\theta = 0$ and $\sigma_\theta = \sqrt{2\lambda}$. The experimentalist would then be conducting measurements on the state

$$\hat{\boldsymbol{\rho}}_\lambda := \frac{1}{\sqrt{4\pi\lambda}} \int_{-\infty}^{\infty} \hat{\mathbf{R}}_z(\theta)|\psi\rangle\langle\psi|\hat{\mathbf{R}}_z^\dagger(\theta) e^{-\frac{\theta^2}{4\lambda}} d\theta. \tag{2.17}$$

Notice that (2.17) has the structure which is necessary and sufficient to generate a quantum map per Theorem 2.1.1. In this case, the Kraus operators consists of the θ dependent family $e^{-\frac{\theta^2}{8\lambda}}\hat{\mathbf{R}}_z(\theta)$, where $\theta \in \mathbb{R}$. Notice that the identity resolution property is easily verified.

$$\frac{1}{\sqrt{4\pi\lambda}} \int_{-\infty}^{\infty} \hat{\mathbf{R}}_z(\theta)\hat{\mathbf{R}}_z^\dagger(\theta) e^{-\frac{\theta^2}{4\lambda}} d\theta = \Big(\frac{1}{\sqrt{4\pi\lambda}} \int_{-\infty}^{\infty} e^{-\frac{\theta^2}{4\lambda}} d\theta\Big)\mathbb{I} = \mathbb{I} \tag{2.18}$$

Furthermore, it can easily be shown that

$$\hat{\boldsymbol{\rho}}_\lambda = |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1| + ab^*e^{-\lambda}|1\rangle\langle 0| + a^*be^{-\lambda}|0\rangle\langle 1| \tag{2.19}$$

which clearly exhibits the damping of the off-diagonal terms, a phenomenon characteristic of decoherence. If λ is large this in turn will mean that the randomness in the environment is large; if λ is arbitrarily large all quantumness (Definition 1.3.4) is wiped out prior to the experimentalist's interacting with S.

2.2.2 Colisional Decoherence as a Quantum Semigroup

In the previous chapter, we studied collisional quantum decoherence (1.104). Clearly, such a model induces time evolution which follows the rules of quantum mechanics. Therefore, (1.104) should be representable in the form required by Theorem 2.1.1. Let us start by rewriting (1.104) in a more suggestive way.

$$\int\int \phi(x)\phi^*(x')e^{-\Lambda t(x-x')^2}|x\rangle\langle x'|dxdx' = \tag{2.20}$$

$$\int\int \phi(x)\phi^*(x')\Big(\int_{\mathbb{R}}\sqrt{\frac{\pi}{t\Lambda}}e^{-\frac{\pi^2}{t\alpha}\xi^2}e^{-2\pi i\xi(x-x')}d\xi\Big)|x\rangle\langle x'|dxdx' \tag{2.21}$$

$$\sqrt{\frac{\pi}{t\Lambda}}\int_{\mathbb{R}}e^{-\frac{\pi^2}{t\alpha}\xi^2}\left(\int\int \phi(x)\phi^*(x')e^{-2\pi i\xi(x-x')}|x\rangle\langle x'|dxdx'\right)d\xi \tag{2.22}$$

$$\sqrt{\frac{\pi}{t\Lambda}}\int_{\mathbb{R}}e^{-\frac{\pi^2}{t\alpha}\xi^2}\left(e^{-2\pi i\xi\hat{\mathbf{X}}}\int\int \phi(x)\phi^*(x')|x\rangle\langle x'|dxdx'e^{2\pi i\xi\hat{\mathbf{X}}}\right)d\xi \tag{2.23}$$

$$\sqrt{\frac{\pi}{t\Lambda}}\int_{\mathbb{R}}e^{-\frac{\pi^2}{t\alpha}\xi^2}\left(e^{-2\pi i\xi\hat{\mathbf{X}}}|\phi\rangle\langle\phi|e^{2\pi i\xi\hat{\mathbf{X}}}\right)d\xi \tag{2.24}$$

$$\int_{\mathbb{R}}\Big(\sqrt[4]{\frac{\pi}{t\Lambda}}e^{-\frac{\pi^2}{2t\alpha}\xi^2}e^{-2\pi i\xi\hat{\mathbf{X}}}\Big)|\phi\rangle\langle\phi|\Big(\sqrt[4]{\frac{\pi}{t\Lambda}}e^{-\frac{\pi^2}{2t\alpha}\xi^2}e^{2\pi i\xi\hat{\mathbf{X}}}\Big)d\xi \tag{2.25}$$

where $|\phi\rangle := \int \phi(x)|x\rangle dx$.

The density operator (2.24) is now confirmed as the non-unitary evolution of the pure state $|\phi\rangle\langle\phi|$ by the quantum map defined by the Kraus operators $\sqrt[4]{\frac{\pi}{t\Lambda}}e^{-\frac{\pi^2}{2t\alpha}\xi^2}e^{-2\pi i\xi\hat{\mathbf{X}}}$. The density operator (2.24) is very much like the density operator we saw in the previous example (2.17) in the sense that they are both averages of unitary dynamics (which result in non-unitary dynamics).

2.2.3 POVM

We have already seen plenty examples of how the environment induces non-unitary dynamics onto some system of interest S. It is an inherent property of measurement processes for the measurer to disturb the system being measured. For classical systems, these disturbances are often inconsequential. However, in quantum mechanics, one works with systems whose observable properties are much more susceptible to disturbance. Measurement is hence naturally described by quantum maps generated by a *Positive Operator Valued Measure* (POVM) defined below. The difference between a POVM and a family of Kraus operators is simply that the Kraus operators generating the POVM are positive semidefinite operators. From the point of view of POVM measurement theory, one may therefore consider environmentally induced decoherence and/or dissipation as a form of measurement performed by the environment, we will call this *environmental monitoring*. A subset of these POVM corresponds to the case of *Projector-Valued Measures* (PVM). The theory of quantum measurement as developed by von Neumann utilizes solely PVM [18].

Definition 2.2.1 (POVM)

Consider an arbitrary Hilbert space $\mathscr{H}$. A POVM is a set of semi-definite operators $\{\hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i\}_i$ acting in $\mathscr{H}$ that sum to the identity operator. i.e.

$$\sum_i \hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i = \mathbb{I}_{\mathscr{H}} \tag{2.26}$$

The POVM may consist of an uncountable set of semi-definite operators as well. In such a case the analogous set of operators, e.g. $\hat{\mathbf{M}}_x$ $(x \in \mathbb{R})$ must meet the same constraint. i.e.

$$\int \hat{\mathbf{M}}_x^\dagger \hat{\mathbf{M}}_x dx = \mathbb{I}_{\mathscr{H}} \tag{2.27}$$

Definition 2.2.2 (Quantum Measurement)

Consider a POVM $\{\hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i\}_i$ acting in some Hilbert space of arbitrary dimension. Furthermore, consider a density operator $\hat{\boldsymbol{\rho}}$ which acts in the same Hilbert space. Given a quantum system in state $\hat{\boldsymbol{\rho}}$, the theory of quantum probability treats the $\hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i$ as events, while the traces $p_i := Tr\{\hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i\}$ are postulated to be the probabilities of the i the event occurring after conducting a measurement on the system designed to read out the events modeled by the POVM. The operators $\hat{\mathbf{M}}_i$ are therefore Kraus operators of a quantum map. If one conducts a measurement on the quantum state $\hat{\boldsymbol{\rho}}$ and the outcome is that which is indexed by i, then the post-measurement state is postulated to be

$$\frac{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger}{Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger\}} \tag{2.28}$$

The state above is the resulting state assuming that one has "read out" the measurement. However, if one does not read out the results of the measurement, what one has is a mixture

$$\sum_i p_i \frac{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger}{Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger\}} \tag{2.29}$$

Given that $p_i = Tr\{\hat{\mathbf{E}}_i \hat{\boldsymbol{\rho}}\}$, the unread state of the system is

$$\sum_i p_i \frac{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger}{Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger\}} = \sum_i Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger\} \frac{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger}{Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger\}} = \sum_i \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_i^\dagger. \tag{2.30}$$

The map (2.30) is clearly a quantum map.

Definition 2.2.3 (von Neumann Measurement)

The theory of *von Neumann Measurement* is just the theory of quantum measurement described above but specialized to the case where the POVM in question are *Projector Valued Measures* (PVM).

2.2.4 Weak Measurement as an Example of POVM Measurement

An observer monitoring the system S may have at their disposal a POVM that measures the spectrum of the position operator $\hat{\mathbf{X}}$, however, due to resolution limitations one expects that any realizable POVM will have finite precision. An example of a viable POVM for estimating the position of some quantum mechanical object can be constructed from Gaussian functions as follwos.

$$\hat{\mathbf{M}}_q = (2\pi\sigma^2)^{-1/4} \int e^{-\frac{(q-x')^2}{4\sigma^2}} |x'\rangle\langle x'| dx'. \tag{2.31}$$

Note that this family of projectors forms a resolution of the identity.

$$\int \hat{\mathbf{M}}_q^\dagger \hat{\mathbf{M}}_q dq = \mathbb{I}.$$

We will refer to the parameter σ^{-1} as the measurement precision. It can be shown that $\lim_{\sigma\to 0} M_q^\dagger M_q = |q\rangle\langle q|$ and $\lim_{\sigma\to\infty} M_q^\dagger M_q = \mathbb{I}$ in the weak sense; these are the max and min precision limits. For finite and/or large values of σ we enter what is known as the weak measurement regime. For a detailed discussion regarding weak measurement theory please see [55]; the interested reader may also look at the theory of Gentle Measurement [11] which generalizes weak measurement. We will be interested in measurements that are approximately non-disturbing, for the case of continuous variables we will adopt the Gentle Measurement Principle [11] to define approximate non-disturbance. i.e. a measurement is non-disturbing if a set of states can be distinguished with high probability; i.e. they can, in principle, be distinguished by a measurement that does not disturb the state. Indeed, for a given infinite dimensional state matrix $\hat{\boldsymbol{\rho}}$ with kernel K, using the POVM $\{\hat{\mathbf{M}}_q\}_q$ we have

$$\int \hat{\mathbf{M}}_q^\dagger \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_q dq = (2\pi\sigma^2)^{-\frac{1}{2}} \int \left\{ \int\int\int e^{-\frac{(q-x)^2}{4\sigma^2}} e^{-\frac{(q-y)^2}{4\sigma^2}} K(x,y)|x\rangle\langle y| dx dy \right\} dq = \tag{2.32}$$

$$= \int\int \left\{ (2\pi\sigma_{ms}^2)^{-\frac{1}{2}} \int e^{-\frac{(q-x)^2}{4\sigma^2}} e^{-\frac{(q-y)^2}{4\sigma^2}} dq \right\} K(x,y)|x\rangle\langle y| dx dy = \tag{2.33}$$

$$= \int\int e^{-\frac{(x-y)^2}{8\sigma^2}} K(x,y)|x\rangle\langle y| dx dy \approx \int\int K(x,y)|x\rangle\langle y| dx dy \quad \sigma\to\infty \tag{2.34}$$

σ need not be infinite for the latter approximation to be valid, it need only be large enough to maintain the decoherence kernel above approximately constant within the support of $K(x,y)$. The larger σ is the weaker the effects of the quantum map generated by the Kraus operator $\hat{\mathbf{M}}_q$ will be.

2.3 Quantifying Disturbance/Noise with the *Trace Distance*

To quantify how much a quantum map disturbs an arbitrary quantum state, we will be using the trace distance. We will define the trace distance shortly, but before this, we need to define the trace norm.

Definition 2.3.1 (Trace Norm)

Let $\mathscr{H}$ be an arbitrary Hilbert space and let $\hat{\boldsymbol{\rho}} \in \mathcal{S}(\mathscr{H})$. Then, the trace norm of $\hat{\boldsymbol{\rho}}$ is defined as

$$\|\hat{\boldsymbol{\rho}}\|_1 := Tr\left\{\sqrt{\hat{\boldsymbol{\rho}}^\dagger \hat{\boldsymbol{\rho}}}\right\} \tag{2.35}$$

Now we define the trace distance.

Definition 2.3.2 (Trace Distance)

Let $\mathscr{H}$ be an arbitrary Hilbert space and let $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\sigma}} \in \mathcal{S}(\mathscr{H})$. The trace distance between the density operators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\sigma}}$ is defined as follows.

$$\mathcal{D}(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}) := \frac{1}{2}\|\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}\|_1 \tag{2.36}$$

Although structurally simple, the trace distance is generally intractable for the same reasons solving the Schrödingers question is (1.1), i.e. the complexity of the eigenvalue problem. To see this more clearly let us look back at equation (1.49) and the preceding discussion. For the case of density operators, the trace is equal to the trace norm so the situation is simpler because we do not need to worry about taking a square root of an operator as required by Definition 2.3.1. In the best of cases, we know the entire spectrum and we are done! However, differences of density operators $\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}$ will not be positive operators in general and will yield further complexity when computing the trace norm. In fact, even if we know the spectral decomposition of $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\sigma}}$ respectively, this does not mean that we will know the spectral decomposition of $\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}$, i.e. not unless $[\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}] = 0$. $\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}$ is also a trace class operator, the spectral theory for compact operators, therefore, tells us that there exists a basis $\{|\phi_i\rangle\}_i$ that diagonalizes $\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}$. With such a basis it can be shown that

$$\|\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}\|_1 = \sum_i |\lambda_i(\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}})|. \tag{2.37}$$

Finding the eigenvectors $\{|\phi\rangle_i\}_i$ is nevertheless a complicated affair, as already discussed in Chapter 2, and should be avoided if possible. The only case for which the trace distance may be easily calculated is when both $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\sigma}}$ are pure states. We present this result as a lemma.

Theorem 2.3.1 (Trace Distance of Two Pure States [9])

Consider two pure states $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$. Their trace distance is the following.

$$\mathcal{D}\Big(|\psi\rangle\langle\psi|,\ |\phi\rangle\langle\phi|\Big) = \sqrt{1 - |\langle\psi|\phi\rangle|^2} \tag{2.38}$$

In the finite-dimensional case, $\mathbf{dim}(\mathscr{H}) < \infty$, it is helpful to note that the trace norm is equivalent any other *Schatten* norm, e.g. the *Hilbert-Schmidt* norm [9].

Definition 2.3.3 (Schattern Norms [47])

Let $\mathscr{H}$ be an arbitrary Hilbert space and let $\hat{\boldsymbol{\rho}} \in \mathcal{S}(\mathscr{H})$. The $p-$Schatten norm is defined as.

$$\|\hat{\boldsymbol{\rho}}\|_p := \sqrt[p]{Tr\Big\{\Big(\sqrt{\hat{\boldsymbol{\rho}}^\dagger\hat{\boldsymbol{\rho}}}\Big)^p\Big\}} \tag{2.39}$$

An interesting property of the *Trace Idelas* [47] corresponding to the respective Schatten norms $\|...\|_p$ is that they satisfy an operator version of *Hölders* inequality. Namely, the following theorem.

Theorem 2.3.2 (Schatten-Hölder)

Let $\mathscr{H}$ be an arbitrary *Hilbert* space and let $\hat{\boldsymbol{\rho}}$,$\hat{\boldsymbol{\sigma}} \in \mathcal{S}(\mathscr{H})$. Then,

$$\|\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\sigma}}\|_1 \leq \|\hat{\boldsymbol{\rho}}\|_q\|\hat{\boldsymbol{\sigma}}\|_p \tag{2.40}$$

whenever $\frac{1}{p} + \frac{1}{q} = 1$.

These Schatten norms induce a metric, and for the finite-dimensional case, they are all equivalent in the following sense. Let $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\sigma}}$ be two finite-dimensional density operators. Then, for all p and q, there exists constants C_1 and C_2 such that

$$C_1\|\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}\|_p \leq \|\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}\|_q \leq C_2\|\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}\|_p. \tag{2.41}$$

The constants depend on p, q and on the dimension of the Hilbert space the operators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\sigma}}$ act in. The Hilbert-Schmidt norm is the case where $p = 2$. i.e. $\|\hat{\boldsymbol{\rho}}\|_2 = \sqrt{Tr\{\hat{\boldsymbol{\rho}}^\dagger\hat{\boldsymbol{\rho}}\}}$. Notice that even if we do not have knowledge of the spectral decomposition of the operator $\hat{\boldsymbol{\rho}}$, we may nevertheless easily compute $\hat{\boldsymbol{\rho}}^\dagger\hat{\boldsymbol{\rho}}$ and subsequently calculate the sum of the diagonal terms in order to obtain the trace; so long as we have a representation of $\hat{\boldsymbol{\rho}}$, it is relatively simple to proceed. This equivalence between the Schatten norms in the finite-dimensional case allows us to estimate trace norms without worrying about the square root in the definition of the trace norm. This approach is unfortunately not viable for the infinite-dimensional case; when dim $(\mathbf{H}) = \infty$ there is no longer equivalence amongst Schatten norms. What is more restrictive is that for an arbitrary trace class operator $\hat{\mathbf{A}}$,

$$\|\hat{\boldsymbol{A}}\| \leq \ldots \leq \|\hat{\mathbf{A}}\|_2 \leq \|\hat{\mathbf{A}}\|_1. \tag{2.42}$$

This means that we have little room to estimate the trace norm; any of the popular trace norm bounds will require us to estimate an equivalently difficult quantity. One of the most popular bounds for the trace distance is the following [9]

$$1 - \sqrt{F(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}})} \leq \frac{1}{2}\|\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}\|_1 \leq \sqrt{1 - F(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}})} \tag{2.43}$$

where $F(\ldots,\ldots)$ is the quantum fidelity defined as follows.

Definition 2.3.4 (Quantum Fidelity)

Let $\mathscr{H}$ be an arbitrary *Hilbert* space and let $\hat{\boldsymbol{\rho}}$,$\hat{\boldsymbol{\sigma}} \in \mathcal{S}(\mathscr{H})$. The quantum fidelity between $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\sigma}}$ is defined as

$$F(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}) := \|\sqrt{\hat{\boldsymbol{\rho}}}\sqrt{\hat{\boldsymbol{\sigma}}}\|_1^2 \tag{2.44}$$

or equivalently

$$F(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}) := Tr\left\{\sqrt{\sqrt{\hat{\boldsymbol{\rho}}}\hat{\boldsymbol{\sigma}}\sqrt{\hat{\boldsymbol{\rho}}}}\right\}^2 \tag{2.45}$$

Notice that the quantum fidelity is no simpler to compute than the trace distance in general. However, when at least one of the states $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\sigma}}$ is pure, the calculation is simpler. i.e.

$$F(|\psi\rangle\langle\psi|, \hat{\boldsymbol{\sigma}}) = |\langle\psi|\hat{\boldsymbol{\sigma}}|\psi\rangle|. \tag{2.46}$$

The fidelity for two pure states is therefore

$$F\Big(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|\Big) = |\langle\psi|\phi\rangle|^2. \tag{2.47}$$

2.3.1 Contractivity of Quantum Maps

An important result exhibiting the effects of quantum maps on a trace distance is

Theorem 2.3.3 (Contractivity of Quantum Maps)

Let $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\sigma}}$ be two density operators and $\mathscr{E}$ a quantum map acting on these states. Then,

$$\mathcal{D}(\mathscr{E}(\hat{\boldsymbol{\rho}}), \mathscr{E}(\hat{\boldsymbol{\sigma}})) \leq \mathcal{D}(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}) \tag{2.48}$$

2.3.2 Disturbance Due to a Quantum Map

How can one tell if a quantum map preserves information? Assume that we have some density operator $\hat{\boldsymbol{\rho}} \in \mathcal{S}(\mathscr{H})$, $\mathscr{H}$ arbitrary, and some quantum map $\mathscr{E} : \mathcal{S}(\mathscr{H}) \to \mathcal{S}(\mathscr{H})$. If the trace distance $\mathcal{D}(\hat{\boldsymbol{\rho}}, \mathscr{E}(\hat{\boldsymbol{\rho}})) = 0$, then this means that the operators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\rho}}$ are indistinguishable!

Proof. Let $\mathcal{D}\big(\hat{\boldsymbol{\rho}},\mathscr{E}(\hat{\boldsymbol{\rho}})\big)=0$ and assume that $\hat{\boldsymbol{\rho}}\neq\mathscr{E}(\hat{\boldsymbol{\rho}})$, i.e. $\hat{\boldsymbol{\rho}}-\mathscr{E}(\hat{\boldsymbol{\rho}})\neq 0$ (the zero operator acting in $\mathscr{H}$). Note that $\mathcal{D}\big(\hat{\boldsymbol{\rho}},\mathscr{E}(\hat{\boldsymbol{\rho}})\big)=0$ implies that

$$\sum_i \big|\lambda_i\big(\hat{\boldsymbol{\rho}}-\mathscr{E}(\hat{\boldsymbol{\rho}})\big)\big|=0 \tag{2.49}$$

This implies that all of the eigenvalues $\lambda_i\big(\hat{\boldsymbol{\rho}}-\mathscr{E}(\hat{\boldsymbol{\rho}})\big)=0$. Owing to the fact that the operator $\hat{\boldsymbol{\rho}}-\mathscr{E}(\hat{\boldsymbol{\rho}})\neq 0$ is trace class, we may write any vector in $\mathscr{H}$ as a linear combination of the eigenvectors of $\hat{\boldsymbol{\rho}}-\mathscr{E}(\hat{\boldsymbol{\rho}})$ (call them $\big\{|\phi_i\rangle\big\}_i$). This means that for any $|\psi\rangle\in\mathscr{H}$

$$\big(\hat{\boldsymbol{\rho}}-\mathscr{E}(\hat{\boldsymbol{\rho}})\big)|\psi\rangle=\big(\hat{\boldsymbol{\rho}}-\mathscr{E}(\hat{\boldsymbol{\rho}})\big)\sum_i c_i|\phi_i\rangle=\sum_i 0\times c_i|\phi_i\rangle=0. \tag{2.50}$$

Hence

$$\hat{\boldsymbol{\rho}}|\psi\rangle=\mathscr{E}(\hat{\boldsymbol{\rho}})|\psi\rangle, \tag{2.51}$$

which is a contradiction! We hence conclude that $\mathcal{D}\big(\hat{\boldsymbol{\rho}},\mathscr{E}(\hat{\boldsymbol{\rho}})\big)=0$ implies that $\hat{\boldsymbol{\rho}}=\mathscr{E}\big(\hat{\boldsymbol{\rho}}\big)$. □

For the finite-dimensional case, one may always utilize the Hilbert-Schmidt distance in lieu of the trace-distance in order to forgo the eigenvalue problem. For the infinite-dimensional case, we will have to get creative.

Disturbing pure states

A relatively simple case of disturbance estimation pertains to the case where the initial state is pure and the quantum map in question is trace-preserving. i.e. we begin with some density operator of the form $|\psi\rangle\langle\psi|$. To estimate how much $\mathscr{E}$ disturbs $|\psi\rangle\langle\psi|$ let us compute the trace distance. Here, using (2.43),

$$\mathcal{D}\big(|\psi\rangle\langle\psi|,\mathscr{E}(|\psi\rangle\langle\psi|)\big)\leq\sqrt{1-F\big(|\psi\rangle\langle\psi|,\mathscr{E}(|\psi\rangle\langle\psi|)\big)}=\sqrt{1-|\langle\psi|\mathscr{E}(|\psi\rangle\langle\psi|)|\psi\rangle|}. \tag{2.52}$$

Proof. All we need to show in order to prove the above is that $F\big(|\psi\rangle\langle\psi|,\mathscr{E}(|\psi\rangle\langle\psi|)\big)=|\langle\psi|\mathscr{E}(|\psi\rangle\langle\psi|)|\psi\rangle|$. To this end, we will use (2.45).

$$F\big(|\psi\rangle\langle\psi|,\mathscr{E}(|\psi\rangle\langle\psi|)\big)=Tr\left\{\sqrt{\sqrt{|\psi\rangle\langle\psi|}\mathscr{E}(|\psi\rangle\langle\psi|)\sqrt{|\psi\rangle\langle\psi|}}\right\}^2= \tag{2.53}$$

$$Tr\left\{\sqrt{|\psi\rangle\langle\psi|\mathscr{E}(|\psi\rangle\langle\psi|)|\psi\rangle\langle\psi|}\right\}^2=\langle\psi|\mathscr{E}(|\psi\rangle\langle\psi|)|\psi\rangle Tr\left\{\sqrt{|\psi\rangle\langle\psi|}\right\}^2=|\langle\psi|\mathscr{E}(|\psi\rangle\langle\psi|)|\psi\rangle|. \tag{2.54}$$

□

Quantum Maps that preserve eigenvectors

Perhaps the simplest case of disturbance estimation is the following. It could happen that the quantum map in question, $\mathscr{E}$, simply rotates the eigensubspaces of some density operator $\hat{\boldsymbol{\rho}}$ of interest.

i.e. say $\hat{\boldsymbol{\rho}}$ has the spectral decomposition

$$\hat{\boldsymbol{\rho}} = \sum_i \alpha_i \big|\phi_i\big\rangle\big\langle\phi_i\big| \tag{2.55}$$

and define the quantum map $\mathscr{E}$ as follows.

$$\mathscr{E}\big(\hat{\boldsymbol{\rho}}\big) = \sum_i \beta_i \big|\phi_i\big\rangle\big\langle\phi_i\big| \tag{2.56}$$

In this case

$$\mathcal{D}\big(\hat{\boldsymbol{\rho}}, \mathscr{E}\big(\hat{\boldsymbol{\rho}}\big)\big) = \sum_i |\alpha_i - \beta_i|. \tag{2.57}$$

Quantum map acting on a general mixed state

If we now consider an arbitrary initial mixed state, i.e. a density operator whose purity may be less than one, things become drastically more difficult. For the infinite-dimensional case, there really is no simple approach. As an example, let us return to the weak measurement case. Consider a situation where the system being monitored S is described by a state undergoing decoherence. i.e. let $\mathscr{E}_t$ be some trace-preserving quantum map inducing decoherence. Let $\hat{\boldsymbol{\rho}}$ be some density operator with kernel K. Assume that the effect of $\mathscr{E}_t$ on $\hat{\boldsymbol{\rho}}$ is the following.

$$\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}\big) = \int\int K(x,y)\Gamma(t(x-y))|x\rangle\langle y|dxdy. \tag{2.58}$$

Where $\Gamma(t(x-y)) \to 0$ as $|t(x-y)| \to \infty$. Now, recall that the weak measurement POVM quantum map in this case can be shown to act as follows

$$\Lambda\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}\big)\big) := \int \hat{\mathbf{M}}_q^\dagger \mathscr{E}_t\big(\hat{\boldsymbol{\rho}}\big)\hat{\mathbf{M}}_q dq = \int\int K(x,y)e^{-\frac{(x-y)^2}{8\sigma^2}}\Gamma(t(x-y))|x\rangle\langle y|dxdy. \tag{2.59}$$

To measure the disturbance induced by the quantum map Λ we must estimate

$$\Big\|\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}\big) - \Lambda\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}\big)\big)\Big\|_1 = \Big\|\int\int K(x,y)\Gamma(t(x-y))(1-e^{-\frac{(x-y)^2}{8\sigma^2}})|x\rangle\langle y|dxdy\Big\|_1 \tag{2.60}$$

To proceed one would need to diagonalize the kernel $K(x,y)\Gamma(t(x-y))(1-e^{-\frac{(x-y)^2}{8\sigma^2}})$ or use numerical techniques to estimate the trace norm of such a kernel, a formidable task.

We now conclude this chapter, having introduced almost all of the theory necessary to build the novel results of this thesis. In the following chapter we will begin to present original results; prior to this we will introduce one final bit of crucial theory, i.e. the theory of *Quantum State Discrimination* (QSD).

Chapter 3

Asymptotic QSD for Countable and Uncountable Mixtures

Quantum State Discrimination (QSD) is the problem of minimizing the error in distinguishing between the elements of a mixture of density operators $\sum_i p_i \hat{\boldsymbol{\rho}}_i$. To understand what is meant by *distinguishing* we must refer back to the concepts of a POVM and quantum measurement discussed in the previous section (Definitions 2.2.1, 2.2.2, and 2.2.3). The QSD optimization problem [66] [72] [25] [22] [23] may now be defined. Let $\mathscr{H}$ be an arbitrary Hilbert space and let $\mathscr{H}$ be the space of density operators acting in $\mathscr{H}$. Given a mixture of density operators,

$$\hat{\boldsymbol{\rho}} = \sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_i \tag{3.1}$$

where $\sum_{i=1}^{N} p_i = 1$, the theory of QSD aims to find a POVM $\{\hat{\mathbf{E}}_l\}_{l=1}^{K} \subset \mathcal{B}(\mathscr{H})$ ($K \geq N$, $\hat{\mathbf{E}}_l = \hat{\mathbf{M}}_l^{\dagger}\hat{\mathbf{M}}_l$ where the $\hat{\mathbf{M}}_l$ are the corresponding kraus operators[31]) which resolves the identity operator of $\mathcal{B}(\mathscr{H})$, and minimizes the object below which we will be referring to as a *probability error*.

$$p_E\big\{\{p_i, \hat{\boldsymbol{\rho}}_i\}_{i=1}^{N}, \{\hat{\mathbf{M}}_l\}_{l=1}^{K}\big\} := 1 - \sum_{i=1}^{N} p_i Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_i^{\dagger}\} \tag{3.2}$$

To see what is the error that (3.2) measures let us consider the unread measurement state (2.30) corresponding to the mixture $\sum_{i=1}^{N} \hat{\boldsymbol{\rho}}_i$ after having undergone a measurement generated by the POVM $\{\hat{\mathbf{M}}_i^{\dagger}\hat{\mathbf{M}}_i\}_{i=1}^{N}$.

$$\sum_{j=1}^{N} \hat{\mathbf{M}}_j \Big(\sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_i\Big) \hat{\mathbf{M}}_j^{\dagger} = \tag{3.3}$$

$$\sum_{i=1}^{N} p_i \hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_i^{\dagger} + \sum_{j=1}^{N} \sum_{i;i\neq j}^{N} p_i \hat{\mathbf{M}}_j \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_j^{\dagger} \tag{3.4}$$

From the definition of POVM provided, it is clear that $Tr\big\{\sum_{j=1}^{N} \hat{\mathbf{M}}_j \Big(\sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_i\Big) \hat{\mathbf{M}}_j^{\dagger}\big\} = 1$, hence from (3.3) and (3.4)

$$1 = \sum_{i=1}^{N} p_i Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_j^\dagger\} + \sum_{j=1}^{N} \sum_{i;i\neq j}^{N} p_i Tr\{\hat{\mathbf{M}}_j \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_j^\dagger\} \tag{3.5}$$

The term $Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_i^\dagger\}$ is the probability that the system modeled by the mixture (3.1) was in the state $\hat{\boldsymbol{\rho}}_i$ given that the outcome of the measurement was the state $\frac{\hat{\mathbf{M}}_i\left(\sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_i\right)\hat{\mathbf{M}}_i^\dagger}{Tr\left\{\hat{\mathbf{M}}_i\left(\sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_i\right)\hat{\mathbf{M}}_i^\dagger\right\}}$, meaning that $\sum_{i=1}^{N} p_i Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_i^\dagger\}$ is the probability that POVM chosen perfectyl *discriminates* between the different $\hat{\boldsymbol{\rho}}_i$ of the mixture (3.1). The term $1 - \sum_{i=1}^{N} p_i Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_i^\dagger\}$, the *probability error*, is hence the probability that the POVM fully fails to discriminate between the elements of the mixture (3.1). Using (3.5) it may also be expressed as

$$\sum_{j=1}^{N} \sum_{i;i\neq j}^{N} p_i Tr\{\hat{\mathbf{M}}_j \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_j^\dagger\}$$

In what follows we will just write p_E in place of $p_E\{\{p_i, \hat{\boldsymbol{\rho}}_i\}_{i=1}^{N}, \{\hat{\mathbf{M}}_l\}_{l=1}^{K}\}$ as a shorthand when the context is clear. With this notation, the QSD optimization problem is the problem of computing the following minimum.

$$\min_{POVM} p_E \tag{3.6}$$

The QSD problem will be called *fully solvable* when $\min_{POVM} p_E = 0$.

In this Chapter we consider an *Asymptotic* QSD: let $\mathscr{E}_{i,n}$ be a family of completely positive linear transformations, mapping density operators $\hat{\boldsymbol{\rho}}$ acting in a Hilbert space $\mathscr{H}_1$ to density operators $\mathscr{E}_{i,n_i}(\hat{\boldsymbol{\rho}})$ acting in a Hilbert space $\mathscr{H}_2$ (equal to $\mathscr{H}_1$ or not). These operators depend on parameters n_i, and we consider the QSD problem for the mixture of the operators $\mathscr{E}_{i,n_i}(\hat{\boldsymbol{\rho}})$ The object of our interest is the asymptotic behavior of the minimal error (3.6) corresponding to this QSD, as some or all $n_i \to \infty$. We will say that the asymptotic QSD problem fully solvable with respect to the parameter n_i when

$$\lim_{|n_i|\to\infty} \min_{POVM} p_E\{\{p_i, \mathscr{E}_{i,n_i}(\hat{\boldsymbol{\rho}})\}_{i=1}^{N}, \{\hat{\mathbf{M}}_l\}_{l=1}^{K}\} = 0 \tag{3.7}$$

the minimization above is understood to be taken for every n_i.

Asymptotic QSD arises naturally in the study of quantum communication, quantum to classical transitions and quantum measurement, just to name a few applications [61][40][22] [26]. As an example consider the case where a state is redundantly prepared by some party A in the state $\hat{\boldsymbol{\rho}}_i$ with probability p_i, n copies of each state being made prior to being communicated to another party. From the perspective of some party B, receiving the state prepared by A, the received state would be a mixture of the following form

$$\sum_i p_i \hat{\boldsymbol{\rho}}_i^{\otimes n} \tag{3.8}$$

In such a case the corresponding maps $\mathscr{E}_{i,n_i}$ have $n_i = n$ for all i and are the map $\hat{\boldsymbol{\rho}}$ to $\hat{\boldsymbol{\rho}}^{\otimes n}$ for all i.

Now, define $\min_{POVMP} p_E(n) := \min_{POVM} p_E\Big\{\{p_i, \hat{\boldsymbol{\rho}}_i^{\otimes n}\}_{i=1}^N, \{\hat{\mathbf{M}}_l\}_{l=1}^K\Big\}$. In [61] it was shown that

$$\frac{1}{3}\xi_{QCB}\Big(\{\hat{\boldsymbol{\rho}}_i\}_{i=1}^N\Big) \leq -\lim_{n\to\infty}\frac{\log\Big(\min_{POVMP} p_E(n)\Big)}{n} \leq \xi_{QCB}\Big(\{\hat{\boldsymbol{\rho}}_i\}_{i=1}^N\Big) \tag{3.9}$$

where ξ_{QCB} is the quantum Chernoff bound for an N mixture, defined as

$$\xi_{QCB}\Big(\{\hat{\boldsymbol{\rho}}_i\}_{i=1}^N\Big) := \min_{i,j}\xi_{QCB}\big(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j\big) \tag{3.10}$$

where

$$\xi_{QCB}\big(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j\big) = -\log\big(\min_{0\leq s\leq 1} Tr\{\hat{\boldsymbol{\rho}}_i^s\hat{\boldsymbol{\rho}}_j^{1-s}\}\big) \tag{3.11}$$

This gives us an idea of how the minimum error probability drops off asymptotically as the redundancy n grows. Indeed as $n\to\infty$ we have $\min_{POVM} p_E \to 0$.

One may also use a Quantum Chernoff-bound-free method for the computation of

$$\lim_{n\to\infty}\min_{POVM} p_E(n)$$

by applying a bound to the minimal probability error in [27] (Theorem 3.2.3 in the following section). Let $\|\hat{\mathbf{A}}\|_1 := Tr\{\sqrt{\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}}}\}$ be the trace norm of the operator $\hat{\mathbf{A}}$ Applying Theorem 3.2.3 we have the following result.

$$\min_{POVM} p_E(n) \leq \sum_i\sum_{j;j\neq i}\sqrt{p_ip_j}\Big\|\sqrt{\hat{\boldsymbol{\rho}}_i^{\otimes n}}\sqrt{\hat{\boldsymbol{\rho}}_j^{\otimes n}}\Big\|_1 = \sum_i\sum_{j;j\neq i}\sqrt{p_ip_j}\Big\|\sqrt{\hat{\boldsymbol{\rho}}_i}\sqrt{\hat{\boldsymbol{\rho}}_j}\Big\|_1^n \tag{3.12}$$

which decays to zero as $n\to\infty$ when $\Big\|\sqrt{\hat{\boldsymbol{\rho}}_i}\sqrt{\hat{\boldsymbol{\rho}}_j}\Big\|_1 < 1$ for all $i, j; j \neq i$. One of the remarkable aspects of such a result is the state-independent nature of the convergence, i.e. so long as the fidelity condition stated in the previous sentence is satisfied, the type of states are irrelevant; e.g. they can be finite or infinite dimensional density operators. QSD will nevertheless be dependent on the states $\hat{\boldsymbol{\rho}}_i$ constituting the mixture in general. Our next example exemplifies this.

More recently, and more pertinently to the theme of this paper, asymptotic QSD has made an appearance in the theory of Spectrum Broadcast Structures (SBS) [40] for *quantum measurement limit* type interactions (see section 2.4 in [31] for a discussion on *quantum measurement limit*). In the SBS framework, a notion of objectivity is introduced which postulates that a specific type of state, called an SBS state [40] [39] [38], will emerge from the asymptotic dynamics. The definition of an SBS state stipulates the calculation of a problem related to that of QSD when proving that a state of interest converges one of these so-called SBS state. The relevant optimization problem is now the *super QSD* problem (SQSD) which is just a simple upper bound of the QSD problem. i.e.

$$SQSD := \min_{POVM} \Big\| 1 - \sum_{i=1}^{N} p_i Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_i^\dagger\} \Big\|_1 \geq \min_{POVM} \Big(1 - \sum_{i=1}^{N} p_i Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_i^\dagger\} \Big) = QSD \quad (3.13)$$

In [38] [39] [40], special attention has been given to SQSD problems of the following form.

$$\min_{POVM} \sum_i p_i \big\| e^{-itx_i \hat{\mathbf{B}}} \hat{\boldsymbol{\rho}} e^{itx_i \hat{\mathbf{B}}} - \hat{\mathbf{M}}_i e^{-itx_i \hat{\mathbf{B}}} \hat{\boldsymbol{\rho}} e^{itx_i \hat{\mathbf{B}}} \hat{\mathbf{M}}_i^\dagger \big\|_1 \quad (3.14)$$

where $x_i \neq x_j$ for all $i, j; j \neq i$ and $\hat{\mathbf{B}}$ is an arbitrary self-adjoint operator; the state being discriminated here is of course $\sum_i p_i e^{-itx_i \hat{\mathbf{B}}} \hat{\boldsymbol{\rho}} e^{itx_i \hat{\mathbf{B}}}$. Such unitarily related mixtures, with a parameter t, arise as a direct consequence of the aforementioned *quantum measurement* limit assumption made in [38][39][40]. The super QSD problem (3.14) is asymptotically *fully solvable* with respect to t only if the associated QSD problem is *fully solvable*. Unlike example (3.8), where the asymptotic *full solvability* of the respective QSD problem was independent of the nature of the states involved, here this is not the case. It is easy to find examples where SQSD optimization problems of the type exhibited in (3.14) does not vanish as $t \to \infty$. e.g, let $\hat{\mathbf{B}}$ be equal to the Pauli matrix $\hat{\boldsymbol{\sigma}}_x$ and let $\hat{\boldsymbol{\rho}} = |z_1\rangle\langle z_1|$ with $\{|z_i\rangle\}_i$ the eigenvectors of the Pauli matrix $\hat{\boldsymbol{\sigma}}_z$. It is easy to show that

$$e^{-itx_i \boldsymbol{\sigma}_x} \hat{\boldsymbol{\rho}} e^{-itx_i \boldsymbol{\sigma}_x} = \quad (3.15)$$

$$\cos^2(tx_i)|z_1\rangle\langle z_1| + \sin^2(tx_i)|z_2\rangle\langle z_2| + \quad (3.16)$$

$$i\cos(tx_i)\sin(tx_i)|z_1\rangle\langle z_2| - i\cos(tx_i)\sin(tx_i)|z_2\rangle\langle z_1| \quad (3.17)$$

Now consider the mixture

$$\sum_{i=1}^{2} p_i e^{-itx_i \boldsymbol{\sigma}_x} \hat{\boldsymbol{\rho}} e^{-itx_i \boldsymbol{\sigma}_x} \quad (3.18)$$

and let $p_i = \frac{1}{2}$ for $i = 1, 2$. An application of a result by Hellström [66], discussed in the next section leads to

$$\min_{POVM} p_E(t) = \frac{1}{2} - \frac{1}{4} \big\| e^{-itx_1 \boldsymbol{\sigma}_x} \hat{\boldsymbol{\rho}} e^{-itx_1 \boldsymbol{\sigma}_x} - e^{-itx_1 \boldsymbol{\sigma}_x} \hat{\boldsymbol{\rho}} e^{-itx_1 \boldsymbol{\sigma}_x} \big\|_1 = \quad (3.19)$$

$$2 \Big| \sqrt{ \big(\cos^2(tx_1) - \cos^2(tx_2)\big)\big(\sin^2(tx_1) - \sin^2(tx_2)\big) - \big(\big|\cos(tx_1)\sin(tx_1) - \cos(tx_2)\sin(tx_2)\big|^2 \big) } \Big| \quad (3.20)$$

Clearly (3.19) does not converge to zero as $t \to \infty$, ergo asymptotic QSD is not *fully solvable* and, by consequence of (3.13), neither is the associated asymptotic SQSD problem.

In this Chapter, we will be focusing on the QSD of unitarily related mixtures (URM); i.e. mixtures of the form $\sum_i p_i \hat{\mathbf{U}}_i(t) \hat{\boldsymbol{\rho}} \hat{\mathbf{U}}_i^\dagger(t)$, where $\hat{\mathbf{U}}_i(t)$ are all unitary operators with the same generator. We will provide a necessary and sufficient condition for the asymptotic *full solvability* of the QSD optimization problem for a broad set of URM; this condition will depend on the spectral properties of the generator of the unitary group characterizing the URM and the nature of the initial state, i.e. the state of the mixture when $t = 0$. In Sections 2, and 3 we will give an overview of some important results from the literature that we shall be using and give further motivation. In section 4 we present one of our main results (Theorem 3.6.2 and Corollary 3.6.1) which gives necessary and sufficient conditions for asymptotic QSD optimization of unitarily related mixtures to be *fully solvable* in a broad setting. In section 5 we shall introduce the optimization problem of Uncountable Quantum State Discrimination (UQSD); a framework that generalizes the problem of QSD. Drawing parallels between QSD and UQSD we prove a necessary condition for UQSD in the unitarily related mixture case to be *fully solvable* in the asymptotic regime with respect to a dynamical parameter t. This condition will again depend only on the spectral properties of the generator of the unitary group characterizing the URM and the nature of the initial state. We conclude this discussion by conjecturing that the analog of Theorem 3.6.2 is true for the UQSD case in the unitarily related mixture setting; we follow this conjecture with some motivation and intuition. Furthermore, we provide examples of QSD and UQSD for a variety of settings and discuss the case where $\hat{\mathbf{B}}$ is finite rank.

3.1 PVM Quantum State Discrimination

Let us consider agin the minimization problem

$$\min_{POVM}\left\{1-\sum_{i=1}^{N}p_iTr\{\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\boldsymbol{M}}_i^{\dagger}\}\right\}. \tag{3.21}$$

The minimization in (3.21) is taken over all POVM which which maps from $\mathcal{S}(\mathscr{H})$ to $\mathcal{S}(\mathscr{H})$. Let us now narrow the set of possible POVM to just those of the projector type. To emphasize this we will change our notation from $\hat{\mathbf{E}}_l$ to $\hat{\mathbf{P}}_l$, and owing to the facts that projectors are self-adjoint and $\hat{\mathbf{P}}^2=\hat{\mathbf{P}}$, the respective measurement operators of $\hat{\mathbf{M}}_l$ will just be $\hat{\mathbf{P}}_l$. Let us rewrite the probability error using this new notation.

$$\min_{PVM}\left\{1-\sum_{i=1}^{N}p_iTr\{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\}\right\}. \tag{3.22}$$

In the above, we have made use of the cyclic property of the trace (1.33). In (3.22) and (3.21) we are minimizing over all PVM and POVM respectively. Furthermore, (3.22) bounds (3.21) from above due to the latter term being a minimization performed on the same objective function as (3.21) but over a smaller set. i.e.

$$\min_{POVM}\left\{1-\sum_{i=1}^{N}p_iTr\{\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\boldsymbol{M}}_i^{\dagger}\}\right\}\leq\min_{PVM}\left\{1-\sum_{i=1}^{N}p_iTr\{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\}\right\} \tag{3.23}$$

In some cases, working with PVM is simpler and suffices. The optimization problems (3.22) and (3.21) will, in general, be intractable; exact solutions exist only in a few specialized cases [22] [23]. The most famous of these cases pertains to the the so-called *Hellström* bound [66]. It is a funny name because it is not a bound. We present the exposition found in [22] below.

Theorem 3.1.1 (Hellström Bound)

Let $\mathscr{H}$ be an arbitrary *Hilbert* space. For any mixture of the form

$$p_1\hat{\boldsymbol{\rho}}_1 + p_2\hat{\boldsymbol{\rho}}_2 \in \mathcal{S}(\mathscr{H}). \tag{3.24}$$

Then,

$$\min_{POVM} p_E\big\{\{p_i, \hat{\boldsymbol{\rho}}_i\}_{i=1}^2, \{\hat{\mathbf{E}}_l\}_{l=1}^2\big\} = \frac{1}{2} - \frac{1}{2}\big\|p_1\hat{\boldsymbol{\rho}}_1 - p_2\hat{\boldsymbol{\rho}}_2\big\|_1. \tag{3.25}$$

where the optimal POVM leading to the minimal error are the projectors $\hat{\mathbf{P}}_+$ and $\hat{\mathbf{P}}_-$ onto the positive and negative subspaces of the operator $p_1\hat{\boldsymbol{\rho}}_1 - p_2\hat{\boldsymbol{\rho}}_2$.

Due to the optimal POVM yielding minimal error in the Hellstr öm bound being a set of projectors, we know that for mixtures of two elements $\sum_{i=1}^2 p_i\hat{\boldsymbol{\rho}}_i$ both PVM QSD and POVM QSD will be equal. i.e.

$$\min_{POVM}\left\{1 - \sum_{i=1}^{2} p_i Tr\big\{\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\boldsymbol{M}}_i^\dagger\big\}\right\} = \frac{1}{2} - \frac{1}{2}\big\|p_1\hat{\boldsymbol{\rho}}_1 - p_2\hat{\boldsymbol{\rho}}_2\big\|_1 = \min_{PVM}\left\{1 - \sum_{i=1}^{2} p_i Tr\big\{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\big\}\right\} \tag{3.26}$$

3.2 Some Useful Theorems

Lower and upper bounds for the probability error in the case of a general mixture exist. Some of the more famous ones are the following. For any mixed quantum states $\{\hat{\boldsymbol{\rho}}_i\}_{i=1}^N$ with respective probabilities $\{p_i\}_i$, the minimum-error probability $\min_{POVM} p_E$ may be bounded as follows for an arbitrary Hilbert space $\mathscr{H}$.

Theorem 3.2.1 (Li and Qiu Bound [25])

$$\min_{POVM} p_E \geq \frac{1}{2}\big(1 - \frac{1}{2(N-1)}\sum_i\sum_{j;j\neq i}\big\|p_i\hat{\boldsymbol{\rho}}_i - p_j\hat{\boldsymbol{\rho}}_j\big\|_1\big) \tag{3.27}$$

Theorem 3.2.2 (Montanaro Bound [72])

$$\min_{POVM} p_E \geq \frac{1}{2}\sum_i\sum_{j;j\neq i} p_ip_jF(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j) \tag{3.28}$$

Theorem 3.2.3 (Knill and Barnum [27])

$$\min_{POVM} p_E \leq \sum_i\sum_{j;j\neq i}\sqrt{p_ip_j}\sqrt{F\big(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j\big)} \tag{3.29}$$

Proven for the case where the underlying Hilbert space is assumed finite-dimensional in [27]. We provide a proof of the same result for the case where $\mathscr{H}$ is infinite-dimensional in subsection 3.2.1.

In [25], necessary and sufficient conditions are introduced in order to arrive at a generalization of the Hellström bound. Unlike the proof by Hellström for the two-state mixture, Qiu does not provide a constructive proof. We, therefore, do not have explicit knowledge of the POVM minimizes p_E although [25] is the work known to the author that makes the greatest advances in this direction. If the inequalities above are not enough for our needs, there also exist convex optimization techniques that may be employed in order to find a global min for the objective fiction p_E (3.2) [22][70] , but we will not be entering this realm for two reasons. Firstly, the mixtures that we shall be studying will be dynamic. i.e. $\sum_i p_i \hat{\boldsymbol{\rho}}_{i,t}$. The mixtures of this type that we will be interested in shall have the following asymptotic properties.

$$F\big(\hat{\boldsymbol{\rho}}_{i,t}, \hat{\boldsymbol{\rho}}_{j,t}\big) \rightarrow 0, \;\; i \neq j, \;\; t \;\; \textit{Large} \tag{3.30}$$

The latter means that their support will become asymptotically non-overlapping very quickly with respect to the relevant time frame. These dynamics will make the aforementioned linear programming techniques quite more complex than the static case; also unnecessary since we will be primarily interested in the asymptotic regime for t. The second reason we will not be utilizing the linear programming schemes mentioned is the dimension of the density operators we shall be working with, i.e. $\mathbf{dim}(\mathscr{H}) = \infty$. In this case, the set of possible PVM or POVM is unwieldy in the sense that it will not be parametrizable like the finite-dimensional case. Although infinite, the set of PVM mapping $\mathcal{S}(\mathbb{C}^2)$ to $\mathcal{S}(\mathbb{C}^2)$ may be parametrized by a finite number of parameters; such is not the case when $\mathbf{dim}(\mathscr{H}) = \infty$.

To conclude this section we shall present some key results pertaining to the quantum fidelity.

Theorem 3.2.4 (Purification dependent version of the Fidelity)

The quantum fidelity $F\big(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}\big)$ is equivalent to the following [9].

$$F\big(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}\big) = \max_{|\chi\rangle} \big|\langle \xi | \chi \rangle\big|^2 \tag{3.31}$$

where $|\psi\rangle$ is any fixed purification of $\hat{\boldsymbol{\rho}}$, and the maximization is over all purifications of $\hat{\boldsymbol{\sigma}}$.

Theorem 3.2.5 (Subconcavity of the Fidelity; A Generalization from the equivalent theorem for singular distributions in [9])

Let $\int p(x)\hat{\boldsymbol{\rho}}_x dx$ and $\int q(x)\hat{\boldsymbol{\sigma}}_x dx$ be two uncountable mixtures (p(x) and q(x) are probability distributions). Then,

$$\sqrt{F\Big(\int p(x)\hat{\boldsymbol{\rho}}_x dx, \int q(x)\hat{\boldsymbol{\sigma}}_x dx\Big)} \geq \int \sqrt{p(x)q(x)} F\big(\hat{\boldsymbol{\rho}}_x, \hat{\boldsymbol{\sigma}}_x\big) dx \tag{3.32}$$

Proof. The proof herein follows the standard methodology seen in [9] Chapter 9 for the countable mixture case. Begin by letting $|\psi_x\rangle$ and $|\sigma_x\rangle$ the purifications of $\hat{\boldsymbol{\rho}}_x$ and $\hat{\boldsymbol{\sigma}}_x$ which satisfy the maximization version of the fidelity; i.e. $F\Big(\hat{\boldsymbol{\rho}}_x, \hat{\boldsymbol{\sigma}}_x\Big) = \big|\langle\psi_x|\phi_x\rangle\big|^2$. We now define

$$|\psi\rangle := \int \sqrt{p(x)}|\psi_x\rangle|x\rangle dx \tag{3.33}$$

$$|\phi\rangle := \int \sqrt{q(x)}|\phi_x\rangle|x\rangle dx. \tag{3.34}$$

$|\psi\rangle$ and $|\phi\rangle$ are purifications of the operators $\int p(x)\hat{\boldsymbol{\rho}}_x dx$ and $\int q(x)\hat{\boldsymbol{\sigma}}_x dx$ where the ancillary space is taken to be $L^2(\mathbb{R})$. Using Theorem 3.2.4 we now have.

$$\sqrt{F\Big(\int p(x)\hat{\boldsymbol{\rho}}_x dx, \int q(x)\hat{\boldsymbol{\sigma}}_x dx\Big)} \geq |\langle\phi|\psi\rangle| = \tag{3.35}$$

$$\left|\int \sqrt{p(x)}\sqrt{q(y)}\langle\psi_x|\phi_y\rangle\langle x|y\rangle dy dx\right| = \left|\int\int \sqrt{p(x)q(x)}\langle\psi_x|\phi_x\rangle dx\right| = \tag{3.36}$$

$$\left|\int \sqrt{p(x)q(x)}\langle\psi_x|\phi_x\rangle dx\right| = \int \sqrt{p(x)q(x)}F\big(\hat{\boldsymbol{\rho}}_x, \hat{\boldsymbol{\sigma}}_x\big) dx \tag{3.37}$$

□

The latter gives us the means by which we may bound fidelities of mixed state from below. There is a useful corollary that follows immediately from Lemma 3.2.5. We present it here below.

Corollary 3.2.1 (Sub-concavity in one argument)

Let $\int p(x)\hat{\boldsymbol{\rho}}_x dx$ be some uncountable mixture, and $\hat{\boldsymbol{\sigma}}$ be some arbitrary density operator (p(x) is a probability distributions). Then,

$$\sqrt{F\Big(\int p(x)\hat{\boldsymbol{\rho}}_x dx, \hat{\boldsymbol{\sigma}}\Big)} \geq \int p(x)F\big(\hat{\boldsymbol{\rho}}_x, \hat{\boldsymbol{\sigma}}\big) dx \tag{3.38}$$

Proof. Note that $\hat{\boldsymbol{\sigma}} = \int p(x)\hat{\boldsymbol{\sigma}} dx$. The proof follows from applying Lemma (3.2.5) to the Fidelity $\sqrt{F\Big(\int p(x)\hat{\boldsymbol{\rho}}_x dx, \int q(x)\hat{\boldsymbol{\sigma}} dx\Big)}$. □

Note that the probability distributions $p(x)$ and $q(x)$ found in the results above pertaining to the quantum fidelity may be *singular*; Dirac measures for example. If $p(x)$ and $q(x)$ are taken to be singular measures then we may obtain the countable versions of the Theorem 3.2.5 and Corollary 3.2.1 ubiquitous in quantum information theory texts such as [9].

Before ending this section, we dedicate a subsection to proving that Theorem 3.2.3 can indeed be generalized to the case of mixtures of infinite dimensional density operators.

3.2.1 Generalizing the Knill-Barnum Bound

Recall that for a mixed state, $\sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_i$ of finite rank density matrices, the QSD problem may be bounded from above as follows (by Theorem 3.2.3).

$$\min_{POVM} \left(1 - \sum_{i=1}^{N} p_i Tr\{\hat{\mathbf{M}}_i \hat{\boldsymbol{\rho}}_j \hat{\mathbf{M}}_i^{\dagger}\}\right) \leq \sum_{i=1}^{N} \sum_{j \neq i}^{N} \sqrt{p_i p_j} \sqrt{F(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j)} \tag{3.39}$$

Before we can use this result in all generality we have to reassure ourselves that this bound will work for the case where the density operators in question, i.e. $\hat{\boldsymbol{\rho}}_i$ are infinite-dimensional. The verification is necessary because the proof of (3.29) in [27] uses techniques that involve inverting linear combinations of the operators $\hat{\boldsymbol{\rho}}_i$, and as is known from functional analysis, infinite-dimensional compact operators are not invertible. Hence, it is necessary to ensure the extension of Theorem3.2.3 to the case of infinite-dimensional density operators. Before we move on to the proof, we will first prove a Lemma that will be useful in proving the upcoming theorem.

Lemma 3.2.1 (Limit Lemma)

Let $\hat{\boldsymbol{\rho}}_{i,d} = \sum_{k=1}^{d} \lambda_{ki} |\psi_{ki}\rangle\langle\psi_{ki}|$ be a rank d approximation of the operator $\hat{\boldsymbol{\rho}}_i$. Then

$$\lim_{d\to\infty} \left\| \sqrt{\hat{\boldsymbol{\rho}}_{i,d}} \sqrt{\hat{\boldsymbol{\rho}}_{j,d}} \right\|_1 = \left\| \sqrt{\hat{\boldsymbol{\rho}}_i} \sqrt{\hat{\boldsymbol{\rho}}_j} \right\|_1 \tag{3.40}$$

Proof.

$$\lim_{d\to\infty} \left| \left\| \sqrt{\hat{\boldsymbol{\rho}}_{i,d}} \sqrt{\hat{\boldsymbol{\rho}}_{j,d}} \right\|_1 - \left\| \sqrt{\hat{\boldsymbol{\rho}}_i} \sqrt{\hat{\boldsymbol{\rho}}_j} \right\|_1 \right| \leq \tag{3.41}$$

$$\lim_{d\to\infty} \left\| \sqrt{\hat{\boldsymbol{\rho}}_{i,d}} \sqrt{\hat{\boldsymbol{\rho}}_{j,d}} - \sqrt{\hat{\boldsymbol{\rho}}_i} \sqrt{\hat{\boldsymbol{\rho}}_j} \right\|_1 \leq \tag{3.42}$$

$$\lim_{d\to\infty} \left\| \sqrt{\hat{\boldsymbol{\rho}}_{i,d}} \sqrt{\hat{\boldsymbol{\rho}}_{j,d}} - \sqrt{\hat{\boldsymbol{\rho}}_i} \sqrt{\hat{\boldsymbol{\rho}}_{j,d}} \right\|_1 + \lim_{d\to\infty} \left\| \sqrt{\hat{\boldsymbol{\rho}}_i} \sqrt{\hat{\boldsymbol{\rho}}_{j,d}} - \sqrt{\hat{\boldsymbol{\rho}}_i} \sqrt{\hat{\boldsymbol{\rho}}_j} \right\|_1 \leq \tag{3.43}$$

$$\lim_{d\to\infty} \left\| \sqrt{\hat{\boldsymbol{\rho}}_{i,d}} - \sqrt{\hat{\boldsymbol{\rho}}_i} \right\|_2 \| \sqrt{\hat{\boldsymbol{\rho}}_{j,d}} \|_2 + \lim_{d\to\infty} \left\| \sqrt{\hat{\boldsymbol{\rho}}_{j,d}} - \sqrt{\hat{\boldsymbol{\rho}}_j} \right\|_2 \| \sqrt{\hat{\boldsymbol{\rho}}_i} \|_2 \leq \tag{3.44}$$

$$\lim_{d\to\infty} \left\| \sqrt{\hat{\boldsymbol{\rho}}_{i,d}} - \sqrt{\hat{\boldsymbol{\rho}}_i} \right\|_2 + \lim_{d\to\infty} \left\| \sqrt{\hat{\boldsymbol{\rho}}_{j,d}} - \sqrt{\hat{\boldsymbol{\rho}}_j} \right\|_2 = \tag{3.45}$$

$$\lim_{d\to\infty} \left\| \sum_{k=d+1} \sqrt{\lambda_{ki}} |\psi_{ki}\rangle\langle\psi_{ki}| \right\|_2 + \lim_{d\to\infty} \left\| \sum_{k=d+1} \sqrt{\lambda_{kj}} |\psi_{kj}\rangle\langle\psi_{kj}| \right\|_2 = \tag{3.46}$$

$$\lim_{d\to\infty} \left(\sqrt{\sum_{k=d+1}^{\infty} \lambda_{ki}} + \sqrt{\sum_{k=d+1}^{\infty} \lambda_{kj}} \right) = \lim_{d\to\infty} \sum_{k=d+1}^{\infty} \left(\sqrt{\lambda_{ki}} + \sqrt{\lambda_{kj}} \right) = 0 \tag{3.47}$$

□

We are now ready to prove that Theorem 3.2.3 may be generalized to the case of mixtures of infinite dimensional density operators. .

Theorem 3.2.6 (Knill and Barnum Bound [27] Generalized)

Let $\sum_{i=1}^{N} p_i \hat{\rho}_i$ be some finite mixture of infinite dimensional density operators, then the Knill Barnum (3.29) bound applies to such a mixture.

Proof. Starting from the optimization problem $\min_{POVM} \sum_{i=1}^{N} \sum_{j;j\neq i}^{M} p_i Tr\{\hat{\mathbf{M}}_j \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_j^\dagger\}$, notice that we may rewrite each $\hat{\boldsymbol{\rho}}_i$ as the limit of a sequence of finite rank operators. To see this, first we diagonalize the $\hat{\boldsymbol{\rho}}_i$. i.e. $\hat{\boldsymbol{\rho}}_i = \sum_{k=1}^{\infty} \lambda_{ki} |\psi_{ki}\rangle\langle\psi_{ki}|$. Where the λ_{ki} are the eigen values of $\hat{\boldsymbol{\rho}}_i$. A d rank approximation of $\hat{\boldsymbol{\rho}}_i$ is therefore $\hat{\boldsymbol{\rho}}_{i,d} := \sum_{k=1}^{d} \lambda_{ki} |\psi_{ki}\rangle\langle\psi_{ki}|$ and indeed

$$\lim_{d\to\infty} \left\| \hat{\boldsymbol{\rho}}_{i,d} - \hat{\boldsymbol{\rho}}_i \right\|_1 = \lim_{d\to\infty} \left\| \sum_{k=d+1}^{\infty} \lambda_{ki} |\psi_{ki}\rangle\langle\psi_{ki}| \right\|_1 \leq \lim_{d\to\infty} \sum_{k=d+1}^{\infty} |\lambda_{ki}| = 0. \tag{3.48}$$

To proceed we must first demonstrate

$$\min_{POVM} \sum_{i=1} \sum_{j;i\neq j} p_i Tr\{\hat{\mathbf{M}}_j \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_j^\dagger\} = \lim_{d\to\infty} \min_{POVM} \sum_{i=1} \sum_{j;i\neq j} p_i Tr\{\hat{\mathbf{M}}_j \hat{\boldsymbol{\rho}}_{i,d} \hat{\mathbf{M}}_j^\dagger\}. \tag{3.49}$$

To show the above we need only show that

$$\lim_{d\to\infty} \left| \min_{POVM} \sum_{i=1} \sum_{j;i\neq j} p_i Tr\{\hat{\mathbf{M}}_j \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_j^\dagger\} - \min_{POVM} \sum_{i=1} \sum_{j;i\neq j} p_i Tr\{\hat{\mathbf{M}}_j \hat{\boldsymbol{\rho}}_{i,d} \hat{\mathbf{M}}_j^\dagger\} \right| = 0. \tag{3.50}$$

We proceed as follows.

$$\left| \min_{POVM} \sum_{i=1} \sum_{j;i\neq j} p_i Tr\{\hat{\mathbf{M}}_j \hat{\boldsymbol{\rho}}_i \hat{\mathbf{M}}_j^\dagger\} - \min_{POVM} \sum_{i=1} \sum_{j;i\neq j} p_i Tr\{\hat{\mathbf{M}}_j \hat{\boldsymbol{\rho}}_{i,d} \hat{\mathbf{M}}_j^\dagger\} \right| \leq \tag{3.51}$$

$$\max_{POVM} \left| \sum_{i=1} \sum_{j;i\neq j} p_i Tr\{\hat{\mathbf{M}}_j (\hat{\boldsymbol{\rho}}_i - \hat{\boldsymbol{\rho}}_{i,d}) \hat{\mathbf{M}}_j^\dagger\} \right| \leq \tag{3.52}$$

$$\max_{POVM} \sum_{i=1} \sum_{j;i\neq j} p_i \left\| \hat{\mathbf{M}}_j (\hat{\boldsymbol{\rho}}_i - \hat{\boldsymbol{\rho}}_{i,d}) \hat{\mathbf{M}}_j^\dagger \right\|_1 \leq \max_{POVM} \sum_{i=1} \sum_{j;j\neq i} p_i \|\hat{\mathbf{M}}_j\|_\infty \left\| (\hat{\boldsymbol{\rho}}_i - \hat{\boldsymbol{\rho}}_{i,d}) \right\|_1 \|\hat{\mathbf{M}}_j^\dagger\|_\infty = \tag{3.53}$$

$$\max_{POVM} \sum_{i=1} \sum_{j;j\neq i} p_i \left\| (\hat{\boldsymbol{\rho}}_i - \hat{\boldsymbol{\rho}}_{i,d}) \right\|_1 = \sum_{i=1} \sum_{j;j\neq i} p_i \left\| (\hat{\boldsymbol{\rho}}_i - \hat{\boldsymbol{\rho}}_{i,d}) \right\|_1 = \tag{3.54}$$

$$\sum_{i=1} \sum_{j;j\neq i} p_i \sum_{k=d+1}^{\infty} |\lambda_{ki}| \leq N \sum_{i} p_i \sum_{k=d+1}^{\infty} |\lambda_{ki}| \tag{3.55}$$

and indeed

$$\lim_{d\to\infty} N \sum_{i=1} p_i \sum_{k=d+1}^{\infty} \lambda_{ki} = \tag{3.56}$$

$$N \sum_{i=1}^{N} p_i \lim_{d\to\infty} \sum_{k=d+1}^{\infty} \lambda_{ki} = N \sum_{j=1}^{N} 0 = 0. \tag{3.57}$$

Therefore,

$$\lim_{d\to\infty}\left|\min_{POVM}\sum_{i=1}^{N}\sum_{j;j\neq i}^{N}p_i Tr\{\hat{\mathbf{M}}_j\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_j^\dagger\}-\min_{POVM}\sum_{i,j;i\neq j}^{N}p_i Tr\{\hat{\mathbf{M}}_j\hat{\boldsymbol{\rho}}_{i,d}\hat{\mathbf{M}}_j^\dagger\}\right|=0 \tag{3.58}$$

which means that

$$\min_{POVM}\sum_{i=1}^{N}\sum_{j;j\neq i}^{N}p_j Tr\{\hat{\mathbf{M}}_j\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_j^\dagger\}=\lim_{d\to\infty}\min_{POVM}\sum_{i=1}^{N}\sum_{j;j\neq i}^{N}p_i Tr\{\hat{\mathbf{M}}_j\hat{\boldsymbol{\rho}}_{i,d}\hat{\mathbf{M}}_j^\dagger\}. \tag{3.59}$$

Let us now introduce a normalization constant $\alpha_{i,d} := Tr\{\hat{\boldsymbol{\rho}}_{i,d}\}$. Using this normalization and the Knill-Barnum bound (3.29) [27] we have

$$\lim_{d\to\infty}\min_{POVM}\sum_{i}^{N}\sum_{j;j\neq i}^{N}p_j\alpha_{i,d}^{-1}Tr\{\hat{\mathbf{M}}_j\alpha_{i,d}\hat{\boldsymbol{\rho}}_{i,d}\hat{\mathbf{M}}_j^\dagger\}\leq \tag{3.60}$$

$$\lim_{d\to\infty}\max_{k}\left(\alpha_{k,d}^{-1}\right)\min_{POVM}\sum_{i=1}^{N}\sum_{j;j\neq i}^{N}p_i Tr\{\hat{\mathbf{M}}_j\alpha_{i,d}\hat{\boldsymbol{\rho}}_{i,d}\hat{\mathbf{M}}_j^\dagger\}\leq \tag{3.61}$$

$$\lim_{d\to\infty}\max_{k}\left(\alpha_{k,d}^{-1}\right)\sum_{i=1}^{N}\sum_{j;j\neq i}^{N}\sqrt{p_ip_j}\sqrt{F(\alpha_{i,d}\hat{\boldsymbol{\rho}}_{i,d},\alpha_{j,d}\hat{\boldsymbol{\rho}}_{j,d})}= \tag{3.62}$$

$$\lim_{d\to\infty}\max_{k}\left(\alpha_{k,d}^{-1}\right)\sum_{i=1}^{N}\sum_{j;j\neq i}^{N}\sqrt{p_ip_j}\left\|\sqrt{\alpha_{i,d}\hat{\boldsymbol{\rho}}_{i,d}}\sqrt{\alpha_{j,d}\hat{\boldsymbol{\rho}}_{j,d}}\right\|_1= \tag{3.63}$$

$$\sum_{i=1}^{N}\sum_{j;j\neq i}^{N}\sqrt{p_ip_j}\lim_{d\to\infty}\max_{k}\left(\alpha_{k,d}^{-1}\right)\sqrt{\alpha_{i,d}\alpha_{j,d}}\left\|\sqrt{\hat{\boldsymbol{\rho}}_{i,d}}\sqrt{\hat{\boldsymbol{\rho}}_{j,d}}\right\|_1= \tag{3.64}$$

$$\sum_{i}^{N}\sum_{j;j\neq i}^{N}\sqrt{p_ip_j}\left(\lim_{d\to\infty}\max_{k}\left(\alpha_{k,d}^{-1}\right)\sqrt{\alpha_{i,d}\alpha_{j,d}}\right)\left(\lim_{d\to\infty}\left\|\sqrt{\hat{\boldsymbol{\rho}}_{i,d}}\sqrt{\hat{\boldsymbol{\rho}}_{j,d}}\right\|_1\right)= \tag{3.65}$$

$$\sum_{i=1}^{N}\sum_{j;j\neq i}^{N}\sqrt{p_ip_j}\sqrt{F(\hat{\boldsymbol{\rho}}_i,\hat{\boldsymbol{\rho}}_j)} \tag{3.66}$$

where we have used lemma 3.2.1 and the fact that $\lim_{d\to}\alpha_{k,d}=1$ for all k in the final equality. □

3.3 Unitarily Related Mixtures

In what follows we will be restricting our attention to a specific type of ensemble $\{p_i,\hat{\boldsymbol{\rho}}_{i,t}\}_{i=1}^N$. Namely, those where

$$\hat{\boldsymbol{\rho}}_{i,t} := e^{-itx_i\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{itx_i\hat{\mathbf{B}}} \tag{3.67}$$

for some self-adjoint operator $\hat{\mathbf{B}}$ and some density operator $\hat{\boldsymbol{\rho}}$ both acting in an arbitrary Hilbert space $\mathscr{H}$. All of the operators $\hat{\boldsymbol{\rho}}_{i,t}$ are unitary evolutions of the density operator $\hat{\boldsymbol{\rho}}$, with the dynamics being generated respectively by the operators $x_i\hat{\mathbf{B}}$. Of particular interest to us will be the case where the operator $\hat{\mathbf{B}}$ has purely continuous spectrum and a non-empty *Rajchman* subspace $\mathscr{H}_{rc}$ with $\hat{\boldsymbol{\rho}} \in \mathcal{S}(\mathscr{H}_{rc})$ (See Section 3.5 of this thesis) [58]; such assumptions will be necessary in order to guarantee (3.30). A particular instance of the latter setup will be the case where $\hat{\mathbf{B}}$ is a quadrature operator [19], aka position or momentum operators (the quadrature operator pairs are a generalization of these). In such a case the entire Hilbert space $L^2(\mathbb{R})$ will be the Rajchman subspace [58]. For starters, let us work in the case where $\hat{\mathbf{B}}$ is a momentum operator; we will then generalize to the case of a general quadrature operator. In section 3.5 we will formalize the concept of the *Rajchman* subspace and use this to prove asymptotic QSD for a broad family of unitarily related mixtures. Before proceeding we present two lemmas.

Lemma 3.3.1 (Trace Lemma)

Let the operator $\hat{\mathbf{B}}$ be a momentum operator (1.20) acting in $\mathcal{D}(\hat{\mathbf{B}}) \subset L^2(\mathbb{R})$. We call $\hat{\mathbf{X}}$ satisfying $[\hat{\mathbf{X}}, \hat{\mathbf{B}}] = i\mathbb{I}$ the conjugate of $\hat{\mathbf{B}}$, aka the position operator. Let $\hat{\boldsymbol{\rho}}$ be some density operator in $\mathcal{S}(L^2(\mathbb{R}))$ with representation

$$\hat{\boldsymbol{\rho}} = \int\int K(x,y)|x\rangle\langle y|dxdy \tag{3.68}$$

in the generalized eigenbais of $\hat{\mathbf{X}}$. Then, it can be shown that

$$e^{-itx_i\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{itx_i\hat{\mathbf{B}}} = \int\int K(x,y)|x+tx_i\rangle\langle y+tx_i|dxdy \tag{3.69}$$

and

$$Tr\{e^{-itx_i\hat{\boldsymbol{B}}}\hat{\boldsymbol{\rho}}e^{itx_i\hat{\mathbf{B}}}\} = \int K(x-tx_i, x-tx_i)dx. \tag{3.70}$$

Proof. The proof hinges on utilizing the following decomposition found in standard quantum textbooks [6].

$$|x\rangle = \int e^{-ibx}|b\rangle db. \left(\int |x\rangle\langle x|dx = \mathbb{I}\right), \tag{3.71}$$

where $|x\rangle$ and $|b\rangle$ are the position and momentum operators' generalized eigenkets respectively. Furthermore,

$$\langle b|x\rangle = \frac{1}{\sqrt{2\pi}}e^{-ibx}. \tag{3.72}$$

We will first prove (3.69) of Lemma 3.3.1. Notice that we may rewrite $\hat{\boldsymbol{\rho}}$, originally expressed in the position generalized basis, in the momentum generalized basis.

$$\hat{\boldsymbol{\rho}} = \int\int K(x,y)|x\rangle\langle y|dxdy = \int\int\int\int K(x,y)|b\rangle\langle b|x\rangle\langle y|b'\rangle\langle b'|dbdb'dxdy = \tag{3.73}$$

$$\int\int\left\{\int\int K(x,y)\langle b|x\rangle\langle y|b'\rangle dxdy\right\}|b\rangle\langle b'|dbdb' = \tag{3.74}$$

$$\int\int\left\{\frac{1}{2\pi}\int\int K(x,y)e^{-ibx}e^{ib'y}dxdy\right\}\big|b\big\rangle\big\langle b'\big|dbdb' = \tag{3.75}$$

$$\int\int\hat{K}(b,b')\big|b\big\rangle\big\langle b'\big|dbdb' \tag{3.76}$$

where $\hat{K}(b,b')$ is the 2-D Fourier transform of the kernel $K(x,y)$. Now,

$$e^{-itx_i\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{itx_i\hat{\mathbf{B}}} = \int\int e^{-itx_ib}e^{itx_ib'}\hat{K}(b,b')\big|b\big\rangle\big\langle b'\big|dbdb' = \tag{3.77}$$

$$\int\int\left\{\frac{1}{2\pi}\int\int e^{-itx_ib}e^{itx_ib'}\hat{K}(b,b')e^{ibx}e^{-ib'y}dbdb'\right\}\big|x\big\rangle\big\langle y\big|dxdy = \tag{3.78}$$

$$\int\int\left\{\frac{1}{2\pi}\int\int e^{i(x-tx_i)b}e^{-i(y-tx_i)b'}\hat{K}(b,b')dbdb'\right\}\big|x\big\rangle\big\langle y\big|dxdy = \tag{3.79}$$

$$\int\int K(x-tx_i,y-tx_i)\big|x\big\rangle\big\langle y\big|dxdy = \int\int K(x,y)|x+tx_i\rangle\langle y+tx_i|dxdy. \tag{3.80}$$

The physicists reading this may simply be used to working with the operator $e^{-itx_i\hat{\mathbf{B}}}$ and immediately identify it as a translation operator. In such a case one may simply take for granted that $e^{-itx_i\hat{\mathbf{B}}}\big|x\big\rangle = \big|x+tx_i\big\rangle$ since it is an elementary result from introductory quantum mechanics.

Finally, proving (3.70) is easier since we may use property (3.69).

$$e^{-itx_i\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{itx_i\hat{\mathbf{B}}} = \int\int K(x,y)\big|x+tx_i\big\rangle\big\langle y+tx_i\big|dxdy = \int\int K(x-tx_i,y-tx_i)\big|x\big\rangle\big\langle y\big|dxdy. \tag{3.81}$$

From here, (3.70) follows directly from the generalization to Theorem 1.3.1 presented in Chapter 2. This generalization may be found in [47] (ADDENDA D). □

Lemma 3.3.2 (Trace Lemma)

Let $\Delta \in \mathbb{R}$ and define $\hat{\mathbf{P}}_\Delta := \int_\Delta |x\rangle\langle x|dx$. The operator $\hat{\mathbf{P}}_\Delta := \int_\Delta |x\rangle\langle x|dx$ is projector acting in the Hilbert space $L^2(\mathbb{R})$. Then, for any $\hat{\boldsymbol{\rho}} \in \mathcal{S}\big(L^2(\mathbb{R})\big)$ with representation $\hat{\boldsymbol{\rho}} = \int\int K(x,y)|x\rangle\langle y|dxdy$, where $K(x,y)$ is the kernel of $\hat{\boldsymbol{\rho}}$,

$$Tr\big\{\hat{\mathbf{P}}_\Delta\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_\Delta\big\} = Tr\big\{\hat{\mathbf{P}}_\Delta\hat{\boldsymbol{\rho}}\big\} = \int_\Delta K(x,x)dx \tag{3.82}$$

Proof.

$$\hat{\mathbf{P}}_\Delta\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_\Delta = \int_\Delta\int_\Delta\int\int K(x,y)\big|w\big\rangle\big\langle w\big|x\big\rangle\big\langle y\big|z\big\rangle\big\langle z\big|dxdydwdz = \tag{3.83}$$

$$\int_\Delta\int_\Delta\int\int\int K(x,y)\delta(w-x)\delta(y-z)\big|w\big\rangle\big\langle z\big|dxdydwdz = \int_\Delta\int_\Delta K(x,y)\big|x\big\rangle\big\langle y\big|dxdy. \tag{3.84}$$

Using, once again, the generalization to Theorem 1.3.1 taking the trace of

$$\int_\Delta\int_\Delta dxdyK(x,y)\big|x\big\rangle\big\langle y\big|$$

simply mounts to integrating along the diagonal of $K(x,y)$ over the set Δ. i.e.

$$Tr\Big\{\int_\Delta\int_\Delta K(x,y)\big|x\big\rangle\big\langle y\big|dxdy\Big\} = \int_\Delta K(x,x)dx. \tag{3.85}$$

Hence, we conclude that

$$Tr\big\{\hat{\mathbf{P}}_\Delta\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_\Delta\big\} = \int_\Delta K(x,x)dx. \tag{3.86}$$

By the cyclicity of the trace, it immediately follows that $Tr\{\hat{\mathbf{P}}_\Delta\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}_\Delta\} = Tr\{\hat{\mathbf{P}}^2_\Delta\hat{\boldsymbol{\rho}}\} = Tr\{\hat{\mathbf{P}}_\Delta\hat{\boldsymbol{\rho}}\}$ and so we have our result. □

We now have the tools for facing the problem at hand, i.e. finding a PVM $\big\{\hat{\mathbf{P}}_l\big\}_l$ such that (3.22) is approximately minimized for the case of the ensemble $\{p_i,\hat{\boldsymbol{\rho}}_{i,t}\}_{i=1}^N$, where again

$$\hat{\boldsymbol{\rho}}_{i,t} := e^{-itx_i\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{itx_i\hat{\mathbf{B}}}. \tag{3.87}$$

where $\hat{\mathbf{B}}$ is a momentum operator and $\hat{\boldsymbol{\rho}}\in\mathcal{S}\big(L^2(\mathbb{R})\big)$. To that end let us partition the real line in the following way.

$$\Delta_{1,t} := \Big(-\infty,\ t\frac{x_1+x_2}{2}\Big) \tag{3.88}$$

$$\Delta_{i,t} := \Big(t\frac{x_{i-1}+x_i}{2},\ t\frac{x_i+x_{i+1}}{2}\Big)\ 1<i<N \tag{3.89}$$

$$\Delta_{N,t} := \Big(t\frac{x_{N-1}+x_N}{2},\infty\Big) \tag{3.90}$$

Indeed $\bigcup_{i=1}^N\Delta_{i,t}=\mathbb{R}$. Letting $\hat{\mathbf{P}}_{\Delta_{i,t}} := \chi_{\Delta_{i,t}}\big(\hat{\mathbf{X}}\big)$, where $\hat{\mathbf{X}}$ is the conjugate operator to $\hat{\mathbf{B}}$, i.e. the corresponding position operator, we have the following.

$$\sum_{i=1}^N P_{\Delta_{i,t}} = \sum_{i=1}^N\int_{\Delta_{i,t}}\big|x\big\rangle\big\langle x\big|dx = \int_{\mathbb{R}}\big|x\big\rangle\big\langle x\big|dx = \mathbb{I}. \tag{3.91}$$

To check if the POVM constructed above $\{\hat{\mathbf{P}}_{\Delta_{i,t}}\}_{i=1}^N$ is efficient in discriminating the mixture $\sum_{i=1}^N p_i\hat{\boldsymbol{\rho}}_{i,t}$, let us compute the $Tr\{\hat{\boldsymbol{\rho}}_{i,t}\hat{\mathbf{P}}_{\Delta_{i,t}}\}$. Computing such traces just involves a simple application of Lemma 3.3.2 and a change of variables. We present the results below and a short computational proof in the following.

$$Tr\{\hat{\mathbf{P}}_{\Delta_{1,t}}\hat{\boldsymbol{\rho}}_{1,t}\} = \int_{\Delta_{1,t}}K(x-tx_1,x-tx_1)dx = \int_{-\infty}^{t\frac{x_2-x_1}{2}}K(x,x)dx \tag{3.92}$$

$$Tr\{\hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\boldsymbol{\rho}}_{i,t}\} = \int_{\Delta_{i,t}}K(x-tx_i,x-tx_i)dx = \int_{t\frac{x_{i-1}-x_i}{2}}^{t\frac{x_{i+1}-x_i}{2}}K(x,x)dx \tag{3.93}$$

$$Tr\{\hat{\mathbf{P}}_{\Delta_{N,t}}\hat{\boldsymbol{\rho}}_{N,t}\} = \int_{\Delta_{N,t}}K(x-tx_N,x-tx_N)dx = \int_{t\frac{x_{N-1}-x_N}{2}}^{\infty}K(x,x)dx \tag{3.94}$$

Proof.

$$Tr\{\hat{\mathbf{P}}_{\Delta_{1,t}}\hat{\boldsymbol{\rho}}_{1,t}\} = \int_{-\infty}^{t\frac{x_1+x_2}{2}} dx K(x-tx_1, x-tx_1) \tag{3.95}$$

Let $u(x) = x - tx_1$, then $du(x) = dx$, $u(t\frac{x_1+x_2}{2}) = t\frac{x_1+x_2}{2} - tx_1 = t\frac{x_2-x_1}{2}$, and furthermore $u(-\infty) = -\infty$. Using this substitutive scheme we may recast (3.92) as $\int_{-\infty}^{t\frac{x_2-x_1}{2}} K(u,u)du$ which is what we set out to prove. In a similar fashion, one may convince oneself that equations (3.93) and (3.94) hold. ☐

For finite $\{x_i\}_{i=1}^N$ all of the above integrals approach 1 asymptotically as $t \to \infty$. The latter implies that (3.22) goes to zero as well. We will state the latter as a theorem.

Theorem 3.3.1

Consider the mixture $\sum_{i=1}^N p_i \hat{\boldsymbol{\rho}}_{i,t}$ where $\hat{\boldsymbol{\rho}}_{i,t} := e^{-itx_i\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{itx_i\hat{\mathbf{B}}}$, $\hat{\mathbf{B}}$ is a momentum operator and $\hat{\boldsymbol{\rho}} \in \mathcal{S}\big(L^2(\mathbb{R})\big)$. Then

$$\min_{PVM}\left\{1 - \sum_{i=1}^N p_i Tr\big\{\hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\big\}\right\} \leq \tag{3.96}$$

$$1 - p_1\int_{-\infty}^{t\frac{x_2-x_1}{2}} K(x,x)dx - \sum_{1<i<N} p_i \int_{t\frac{x_{i-1}-x_i}{2}}^{t\frac{x_{i+1}-x_i}{2}} K(x,x)dx - p_N\int_{t\frac{x_{N-1}-x_N}{2}}^{\infty} K(x,x)dx \tag{3.97}$$

For $t \to \infty$ the above upper bound approaches $1 - \sum_{i=1}^N p_i = 1 - 1 = 0$ achieving optimal QSD.

We have focused on the Hilbert space $L^2\big(\mathbb{R}\big)$ but these results may be easily extended to the case of n dimensional square-integrable functions by the same methodology.

3.4 More Unitarily Related Mixtures

Hitherto we constrained ourselves to a mixture of the form

$$\sum_{k=1}^N \hat{\boldsymbol{\rho}}_{k,t} := \sum_{k=1}^N e^{-itx_k\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{itx_k\hat{\mathbf{B}}} \tag{3.98}$$

where $\hat{\mathbf{B}}$ was taken to be the momentum operator and $\hat{\boldsymbol{\rho}} \in \mathcal{S}\big(L^2(\mathbb{R}^2)\big)$. In this section, we will gradually move toward studying more general mixtures of the unitarily related type and their respective QSD problem. Firstly, we will study the case where the unitary evolution is generated by parametrized *Displacemet* [19] operators, a generalization of the momentum operator; we will analyze such a case for a simple *coherent* state as our $t = 0$ density operator $\hat{\boldsymbol{\rho}}$, then we shall generalize to an arbitrary state.

3.4.1 Quadrature Operators

A generalization of the momentum-position conjugate pair of operators may be arrived at by recalling that the position and momentum operators $\hat{\mathbf{P}}$ and $\hat{\mathbf{X}}$ may be written in terms of ladder operators. We present them below in the ladder operator form discussed in (1.93) (1.94), including units.

$$\hat{\mathbf{X}} = \sqrt{\frac{1}{2m\omega}}(\hat{\mathbf{a}} + \hat{\mathbf{a}}^\dagger) \tag{3.99}$$

$$\hat{\mathbf{P}} = -i\sqrt{\frac{m\omega}{2}}(\hat{\mathbf{a}} - \hat{\mathbf{a}}^\dagger) \tag{3.100}$$

$$[\hat{\mathbf{X}},\ \hat{\mathbf{P}}] = i\mathbb{I} \tag{3.101}$$

where we remind the reader that the operators $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^\dagger$ may be understood by their action on number states $\big|n\big\rangle$ ($\big\langle x\big|n\big\rangle$, are the Hermite functions introduced in (1.25)). i.e.

$$\hat{\mathbf{a}}^\dagger\big|n\big\rangle = \sqrt{n+1}\big|n+1\big\rangle \tag{3.102}$$

$$\hat{\mathbf{a}}\big|n\big\rangle = \sqrt{n}\big|n\big\rangle. \tag{3.103}$$

For the following study, the unit carrying terms such as the angular frequency ω, and mass m will be immaterial. The pair of equations (3.99) and (3.100) can be traded in for

$$\hat{\mathbf{X}} = \frac{1}{\sqrt{2}}(\hat{\mathbf{a}} + \hat{\mathbf{a}}^\dagger) \tag{3.104}$$

$$\hat{\mathbf{P}} = -\frac{i}{\sqrt{2}}(\hat{\mathbf{a}} - \hat{\mathbf{a}}^\dagger) \tag{3.105}$$

without losing anything of mathematical importance. It can be shown that the commutation relation (3.101) will remain unchanged [19]. We now present the generalization of the position-momentum pair (3.104) (3.105).

$$\hat{\mathbf{Q}}_\phi = \frac{1}{\sqrt{2}}(e^{i\,\phi}\,\hat{\mathbf{a}} + e^{-i\,\phi}\,\hat{\mathbf{a}}^\dagger) \tag{3.106}$$

$$\hat{\mathbf{P}}_\phi = \frac{-i}{\sqrt{2}}\,(e^{i\,\phi}\,\hat{\mathbf{a}} - e^{-i\,\phi}\,\hat{\mathbf{a}}^\dagger). \tag{3.107}$$

Notice that the latter pair of operators reduces to the former when $\phi = 1$. It can be shown that canonical commutation relations will remain the same as those for the usual position and momentum operators, independent of ϕ, i.e.

$$[\hat{\mathbf{Q}}_\phi,\ \hat{\mathbf{P}}_\phi] = i\mathbb{I}, \tag{3.108}$$

meaning that the algebras $\left\{\mathbb{I}, \hat{\mathbf{Q}}_\phi, \hat{\mathbf{P}}_\phi\right\}$ are equivalent (via isomorphism) to the algebra $\left\{\mathbb{I}, \hat{\mathbf{X}}, \hat{\mathbf{P}}\right\}$ for all $\phi \in \mathbb{R}$. It is interesting to note that for any $\phi \in \mathbb{R}$

$$[\hat{\mathbf{Q}}_\phi, \hat{\mathbf{Q}}_{\phi+\frac{\pi}{2}}] = i\mathbb{I}, \tag{3.109}$$

meaning that the quadrature momenta are $\frac{\pi}{2}$ rotations of the respective conjugate operator in some sense.

3.4.2 Unitarily Related Countable Mixtures. The case for Coherent States

Let $\hat{\mathbf{P}}_\phi$ be some quadrature momentum and let $\hat{\boldsymbol{\rho}} = |0\rangle\langle 0|$ (the vacuum state of the *Fock* basis, the underlying Hilbert space here is $L^2(\mathbb{R})$ but could be generalized to $L^2(\mathbb{R}^n)$ with ease). Let us tackle the QSD problem for the mixture

$$\sum_{k=1}^{N} p_k e^{-itx_k\hat{\mathbf{P}}_\phi}|0\rangle\langle 0|e^{itx_k\hat{\mathbf{P}}_\phi} \tag{3.110}$$

where all of the x_i are unequal to each other. It will be instructive to reorganize the way in which the unitary operator $e^{-itx_k\hat{\mathbf{P}}_\phi}$ is expressed. i.e.

$$e^{-itx_k\hat{\mathbf{P}}_\phi} = e^{-itx_k\left(\frac{-i}{\sqrt{2}}(e^{i\phi}\hat{\mathbf{a}}-e^{-i\phi}\hat{\mathbf{a}}^\dagger)\right)} = e^{\frac{-tx_k}{\sqrt{2}}(e^{i\phi}\hat{\mathbf{a}}-e^{-i\phi}\hat{\mathbf{a}}^\dagger)} =$$

$$= e^{\alpha_k(t)\hat{\mathbf{a}}^\dagger-\alpha_k(t)^*\hat{\mathbf{a}}}, \;\; where \quad \alpha_k(t) := \frac{tx_k e^{-i\phi}}{\sqrt{2}}.$$

The unitary operator $e^{\alpha_k(t)\hat{\mathbf{a}}^\dagger-\alpha_k(t)^*\hat{\mathbf{a}}}$ will henceforth be denoted $\hat{\mathbf{D}}(\alpha_k(t))$. This operator is known as the *displacement* operator [6] [19]. Some of its most useful properties are presented below. Let $|\alpha\rangle$ be some coherent state [19] (also see Chapter 7 of [6]) , then

$$\hat{\mathbf{D}}(\beta)\hat{\mathbf{D}}(\gamma) = \hat{\mathbf{D}}(\beta+\gamma)e^{(\beta\gamma^*-\beta^*\gamma)/2} \tag{3.111}$$

$$\hat{\mathbf{D}}(\beta)|\alpha\rangle = e^{(\beta\alpha^*-\beta^*\alpha)/2}|\beta+\alpha\rangle \tag{3.112}$$

$$\hat{\mathbf{D}}(\alpha)|0\rangle = |\alpha\rangle \tag{3.113}$$

The final two properties above give merit to the name "displacement operator" of $\hat{\mathbf{D}}(\beta)$. Using the latter properties, (3.110) may be shown to be a mixture of coherent states.

Proof.

$$\sum_{k=1}^{N} p_k e^{-itx_k\hat{\mathbf{P}}_\phi}|0\rangle\langle 0|e^{itx_k\hat{\hat{\mathbf{P}}}_\phi} = \sum_{k=1}^{N} p_k\hat{\mathbf{D}}(\alpha_k(t))|0\rangle\langle 0|\hat{\mathbf{D}}^\dagger(\alpha_k(t)) = \tag{3.114}$$

$$\sum_{k=1}^{N} p_k|\alpha_k(t)\rangle\langle\alpha_k(t)| \tag{3.115}$$

□

It can be shown that $|\langle\beta|\alpha\rangle|^2 = e^{-|\beta-\alpha|^2}$ [19] for two arbitrary coherent states $|\alpha\rangle$ and $|\beta\rangle$. The latter implies that $F\Big(|\alpha_k(t)\rangle\langle\alpha_k(t)|, |\alpha_p(t)\rangle\langle\alpha_p(t)|\Big) = e^{-\frac{t^2}{2}|x_k-x_p|^2}$, using (2.47).

Proof.

$$F\big(|\alpha_k(t)\rangle\langle\alpha_k(t)|, |\alpha_p(t)\rangle\langle\alpha_p(t)|\big) \tag{3.116}$$

$$|\langle\alpha_k|\alpha_p\rangle|^2 = e^{-|\alpha_k(t)-\alpha_p(t)|^2} = \tag{3.117}$$

$$e^{-|\frac{tx_k e^{-i\phi}}{\sqrt{2}}-\frac{tx_p e^{-i\phi}}{\sqrt{2}}|^2} = e^{-\frac{t^2}{2}|x_k-x_p|^2}. \tag{3.118}$$

□

Due to the latter it is indeed clear that $F\Big(\big|\alpha_k(t)\big\rangle\big\langle\alpha_k(t)\big|, \big|\alpha_p(t)\big\rangle\big\langle\alpha_p(t)\big|\Big) \to 0$ as $t \to \infty$ for $k \neq p$. This in turn means that the mixture (3.110) will be asymptotically discriminable. i.e. using the generalization of Theorem 3.29, Theorem 3.2.6 we have

$$\lim_{t\to\infty} \min_{POVM} \left\{1 - \sum_{i=1}^{N} p_i Tr\Big\{\hat{\mathbf{M}}_i \big|\alpha_i(t)\big\rangle\big\langle\alpha_i(t)\big| \hat{\boldsymbol{M}}_i^{\dagger}\Big\}\right\} \leq \tag{3.119}$$

$$\lim_{t\to\infty} \sum_{i=1}^{N} \sum_{j;j\neq i}^{N} \sqrt{p_i p_j} \sqrt{F\Big(\big|\alpha_i(t)\big\rangle\big\langle\alpha_i(t)\big|, \big|\alpha_j(t)\big\rangle\big\langle\alpha_j(t)\big|\Big)} = \tag{3.120}$$

$$\lim_{t\to\infty} \sum_{i=1}^{N} \sum_{j;j\neq i}^{N} \sqrt{p_i p_j} e^{-\frac{t^2}{4}|x_i - x_j|^2} = \lim_{t\to\infty} \sum_{i=1}^{N} \sum_{j;j\neq i}^{N} \sqrt{p_i p_j} \lim_{t\to\infty} e^{-\frac{t^2}{4}|x_i - x_j|^2} = 0 \tag{3.121}$$

3.4.3 Unitarily Related Countable Mixtures of Arbitrary Displaced Pure Initial States

Continuing the work of the previous subsection, we proceed with a generalization of the countable mixture (3.110). In this case, the initial state $\big|\psi\big\rangle\big\langle\psi\big|$ will be taken to be pure, but otherwise arbitrary. i.e.

$$\sum_{k=1}^{N} p_k \hat{\boldsymbol{\rho}}_{k,t} := \sum_{k=1}^{N} p_k \hat{\mathbf{D}}(\alpha_k(t)) \big|\psi\big\rangle\big\langle\psi\big| \hat{\mathbf{D}}^{\dagger}(\alpha_k(t)) \tag{3.122}$$

where again, $\alpha_k(t) = \frac{t x_k e^{-i\phi}}{\sqrt{2}}$. We will prove that $F\big(\hat{\boldsymbol{\rho}}_{k,t}, \hat{\boldsymbol{\rho}}_{p,t}\big) \to 0$ as $t \to \infty$ yet again via a bound characterizing the exponential decay as $t \to \infty$; the associated QSD problem will hence be solvable in the asymptotic limit $t \to \infty$. This will consequently prove that under such a generalization the asymptotic QSD problem is still solvable. Before we begin the proof, we must introduce a few definitions.

Definition 3.4.1 (Segal-Bargmann Space)

The *Segal-Bargmann* space is the space of holomorphic functions $F(z)$ in n complex variables satisfying the square-integrability condition:

$$\|F(z)\|_{SB}^2 := \pi^{-n} \int_{\mathbb{C}^n} |F(z)|^2 \exp(-|z|^2)\, dz < \infty \tag{3.123}$$

where dz denotes the 2^n-dimensional Lebesgue measure on $\mathbb{C}^n$. It is a Hilbert space with respect to the associated inner product:

$$\langle F(z) \mid G(z)\rangle = \pi^{-n} \int_{\mathbb{C}^n} \overline{F(z)} G(z) \exp(-|z|^2)\, dz. \tag{3.124}$$

Lemma 3.4.1 (Simple Lemma)

If the Holomorphic function $F(z)$ is in the *Segal-Bargmann* space, then so is the function $F(2z)$.

Proof. This follows from the fact that both 2 and $F(z)$ are in the *Segal-Bargmann* space and are therefore measurable with respect to the measure $e^{-|\alpha|^2}d^2\alpha$, and from the fact that the composition of measurable functions is measurable. □

We now prove that $F\big(\hat{\boldsymbol{\rho}}_{k,t}, \hat{\boldsymbol{\rho}}_{p,t}\big) \to 0$ as $t \to \infty$, but first state it as a Claim.

Claim 3.4.1 (Convergence Claim)

$$F\big(\hat{\boldsymbol{\rho}}_{k,t}, \hat{\boldsymbol{\rho}}_{p,t}\big) \to 0 \;\; as \;\; t \to \infty \tag{3.125}$$

Proof.

$$\hat{\mathbf{D}}^{\dagger}(\alpha_p(t))\hat{\mathbf{D}}(\alpha_k(t)) = \frac{1}{\pi}\int \hat{\mathbf{D}}^{\dagger}(\alpha_p(t))\big|\alpha\big\rangle\big\langle\alpha\big|\hat{\mathbf{D}}(\alpha_k(t))d^2\alpha = \tag{3.126}$$

$$\frac{1}{\pi}\int \big|\alpha - \alpha_p(t)\big\rangle\big\langle\alpha - \alpha_k(t)\big|d^2\alpha \tag{3.127}$$

Where we used the resolution of the identity afforded by the overcomplete set of coherent states, i.e. $\frac{1}{\pi}\int \big|\alpha\big\rangle\big\langle\alpha\big|d^2\alpha = \mathbb{I}$ [19]. Hence,

$$\big\langle\psi\big|\hat{\mathbf{D}}^{\dagger}(\alpha_p(t))\hat{\mathbf{D}}(\alpha_k(t))\big|\psi\big\rangle = \frac{1}{\pi}\int \big\langle\psi\big|\alpha - \alpha_p(t)\big\rangle\big\langle\alpha - \alpha_k(t)\big|\psi\big\rangle d^2\alpha. \tag{3.128}$$

The functions

$$e^{-\frac{1}{2}|\alpha-\alpha_k|^2}\Psi_{SB}((\alpha-\alpha_k)^*) := e^{-\frac{1}{2}|\alpha-\alpha_k|^2}\sum_{n=0}^{\infty}\big\langle n\big|\psi\big\rangle\frac{((\alpha-\alpha_k)^*)^n}{\sqrt{n!}} = \big\langle\alpha-\alpha_k\big|\psi\big\rangle \tag{3.129}$$

where $\Psi_{SB}(\alpha^*)$ belongs to the *Segal-Bargman* space for an arbitrary $\big|\psi\big\rangle$ in the appropriate space of square-integrable functions. We may therefore write

$$|\big\langle\psi\big|\hat{\mathbf{D}}^{\dagger}(\alpha_p(t))\hat{\mathbf{D}}(\alpha_k(t))\big|\psi\big\rangle| = \left|\frac{1}{\pi}\int e^{-\frac{1}{2}|\alpha-\alpha_p|^2}\Psi^*_{SB}((\alpha-\alpha_p)^*)e^{-\frac{1}{2}|\alpha-\alpha_k|^2}\Psi_{SB}((\alpha-\alpha_k)^*)d^2\alpha\right| \leq \tag{3.130}$$

$$\frac{1}{\pi}\int e^{-\frac{1}{2}|\alpha-\alpha_p|^2}e^{-\frac{1}{2}|\alpha-\alpha_k|^2}|\Psi_{SB}((\alpha-\alpha_p)^*)||\Psi_{SB}((\alpha-\alpha_k)^*)|d^2\alpha = \tag{3.131}$$

$$\frac{1}{\pi}\int e^{-\frac{1}{4}|\alpha-\alpha_p|^2}e^{-\frac{1}{4}|\alpha-\alpha_k|^2}e^{-\frac{1}{4}|\alpha-\alpha_p|^2}e^{-\frac{1}{4}|\alpha-\alpha_k|^2}|\Psi_{SB}((\alpha-\alpha_p)^*)||\Psi_{SB}((\alpha-\alpha_k)^*)|d^2\alpha \leq \tag{3.132}$$

$$\frac{1}{\pi}\left(\int e^{-\frac{1}{2}|\alpha-\alpha_p|^2}e^{-\frac{1}{2}|\alpha-\alpha_k|^2}d^2\alpha\right)\left(\int e^{-\frac{1}{2}|\alpha-\alpha_p|^2}e^{-\frac{1}{2}|\alpha-\alpha_k|^2}|\Psi_{SB}((\alpha-\alpha_p)^*)|^2|\Psi_{SB}((\alpha-\alpha_k)^*)|^2d^2\alpha\right) \leq \tag{3.133}$$

$$\frac{1}{\pi}\left(\int e^{-\frac{1}{2}|\alpha-\alpha_p|^2}e^{-\frac{1}{2}|\alpha-\alpha_k|^2}d^2\alpha\right)\left\|e^{-\frac{1}{2}|\alpha|^2}|\Psi_{SB}(\alpha^*)|^2\right\|_{L_\infty(\mathbb{C})}\left\|e^{-\frac{1}{2}|\alpha|^2}|\Psi_{SB}(\alpha^*)|^2\right\|_{L^1(\mathbb{C})} = \tag{3.134}$$

$$\frac{\pi}{2}\left\|e^{-\frac{1}{2}|\alpha|^2}|\Psi_{SB}((\alpha)^*)|^2\right\|_{L_\infty}\left\|\Psi_{SB}(2\alpha^*)\right\|_{SB}{}^2\left(e^{-\frac{1}{2}\left(\Re(\alpha_p)-\Re(\alpha_k)\right)^2}\right)\left(e^{-\frac{1}{2}\left(\Im(\alpha_p)-\Re(\Im_k)\right)^2}\right) = \tag{3.135}$$

$$\frac{\pi}{2}\left\|e^{-\frac{1}{2}|\alpha|^2}|\Psi_{SB}(\alpha^*)|^2\right\|_{L_\infty}\left\|\Psi_{SB}(2\alpha^*)\right\|_{SB}\left(e^{-\frac{\Re(e^{-i\phi})^2t^2}{4}\left(x_p-x_k\right)^2}\right)\left(e^{-\frac{\Im(e^{-i\phi})^2t^2}{4}\left(x_p-x_k\right)^2}\right)= \tag{3.136}$$

$$\frac{\pi}{2}\left\|e^{-|\alpha|^2}|\Psi_{SB}(2\alpha^*)|^2\right\|_{L_\infty}\left\|\Psi_{SB}(2\alpha^*)\right\|_{SB}e^{-\frac{t^2}{4}\left(x_p-x_k\right)^2} \tag{3.137}$$

The fucntion $\Psi_{SB}(\alpha^*)$ is in the *Segal-Bargmann* space, hence its accociated *Segal-Bargmann* norm exists and the fucntion $|e^{-|\alpha|^2}|\Psi_{SB}(2\alpha^*)|^2$ is bounded and $\left\|\Psi_{SB}(2\alpha^*)\right\|_{SB}$ is just a constant by Lemma 3.4.1. We, therefore, conclude that for $k \neq p$

$$F\big(\hat{\boldsymbol{\rho}}_p(t), \hat{\boldsymbol{\rho}}_k(t)\big) = \left|\langle\psi\middle|\hat{\mathbf{D}}^\dagger(\alpha_p(t))\hat{\mathbf{D}}(\alpha_k(t))|\psi\rangle\right|^2 \leq \tag{3.138}$$

$$\frac{\pi}{2}\left\|e^{-|\alpha|^2}|\Psi_{SB}(2\alpha^*)|^2\right\|_{L_\infty}\left\|\Psi_{SB}(2\alpha^*)\right\|_{SB}e^{-\frac{t^2}{4}\left(x_p-x_k\right)^2} \tag{3.139}$$

which goes to zero as $t \to 0$ due to the Gaussian term. □

The associated QSD problem is therefore progressively more solvable as $t \to \infty$. The benefit of the approach taken in Claim 3.4.1 is the ability to bound the fidelity terms by the Gaussian term. The weakness of Claim 3.4.1 is that it only gives a handle on unitarily related mixtures that are generated by displacement operators. If more general unitarily related mixtures are to be studied, we will need to introduce new ideas.

Another approach.

We have used some tools from quantum optics to argue the asymptotic decay of the fidelity of the states $\hat{\boldsymbol{\rho}}_p$ and $\hat{\boldsymbol{\rho}}_k$ when $k \neq p$. However, we could also use the spectrum of the displacement operators $\hat{\mathbf{D}}(\alpha_k(t))$ in order to compute $F\big(\hat{\boldsymbol{\rho}}_p, \hat{\boldsymbol{\rho}}_k\big)$ directly as opposed to bounding it, like we did in proving Claim 3.4.1. Recall that $\hat{\mathbf{D}}(\alpha_k(t)) = e^{-itx_k\hat{\mathbf{P}}_\phi}$ for $\alpha_k(t) = \frac{tx_ke^{-i\phi}}{\sqrt{2}}$. The quadrature momentum $\hat{\mathbf{P}}_\phi$ has purely absolutely continuous spectrum [3]. We will denote its generalized eigenvectors as $|p_\phi\rangle$ (Dirac delta distributions), where the action of $\hat{p}_\phi$ on its generalized eigenvectors is, of course, the following.

$$\hat{\mathbf{P}}_\phi|p_\phi\rangle = p_\phi|p_\phi\rangle. \tag{3.140}$$

We may now compute $\hat{\mathbf{D}}(\alpha_k(t))\big|\psi\big\rangle = e^{-itx_k\hat{\mathbf{P}}_\phi}\big|\psi\big\rangle = \int e^{-itx_kp_\phi}\psi(p_\phi)\big|p_\phi\big\rangle dp_\phi$, which leads to the equation

$$F\big(\hat{\boldsymbol{\rho}}_p, \hat{\boldsymbol{\rho}}_k\big) = \left|\int|\psi(p_\phi)|^2e^{-it(x_k-x_p)p_\phi}dp_\phi\right|^2 \tag{3.141}$$

which is a continuous function decaying asymptotically as $t \to \infty$ due to the *Reimann-Lebesgue* lemma since $|\psi(p_\phi)|^2$ is in $L^1(\mathbb{R})$. The hurdle of this approach is the necessity to compute the Fourier transform of the function $|\psi(p_\phi)|^2$, which could be a daunting task. Both approaches, the one leading to (3.141) and the one leading to the bound (3.139) have their merit.

3.5 Countable Mixtures of Unitarily Related Families

In this section, we will be restricting our attention to a specific type of ensemble $\{p_i, \hat{\boldsymbol{\rho}}_{i,t}\}_{i=1}^{N}$. Namely, those where

$$\hat{\boldsymbol{\rho}}_{i,t} := e^{-itx_i\hat{\mathbf{B}}}|\psi\rangle\langle\psi|e^{itx_i\hat{\mathbf{B}}} \tag{3.142}$$

for some self-adjoint operator $\hat{\mathbf{B}}$ and some pure density operator $|\psi\rangle\langle\psi|$ (i.e. $(|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|$) both acting in an arbitrary Hilbert space $\mathscr{H}$. All of the operators $\hat{\boldsymbol{\rho}}_{i,t}$ are unitary evolutions of the density operator $\hat{\boldsymbol{\rho}}$, with the dynamics being generated respectively by the operators $x_i\hat{\mathbf{B}}$. We will see that the asymptotic discriminability of the mixture $\sum_{i=1}^{N} p_i\hat{\boldsymbol{\rho}}_{i,t}$ will depend on the spectral properties of the operators $\hat{\mathbf{B}}_k$ and the nature of the pure state $|\psi\rangle\langle\psi|$. Using Theorem 3.2.3 we have the following QSD estimate.

$$\min_{POVM}\Big(1-\sum_{i=1}^{N}p_iTr\{\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_{i,t}\hat{\mathbf{M}}_i^{\dagger}\}\Big) \leq \sum_i\sum_{j;j\neq i}\sqrt{p_ip_j}\sqrt{F(\hat{\boldsymbol{\rho}}_{i,t},\hat{\boldsymbol{\rho}}_{j,t})} = \tag{3.143}$$

$$\sum_i\sum_{j;j\neq i}\sqrt{p_ip_j}\big|\langle\psi|e^{-it(x_j-x_i)\hat{\mathbf{B}}}|\psi\rangle\big| \tag{3.144}$$

If *fully solvable* asymptotic QSD is desired, then it is a necessary and sufficient condition that the elements $\big|\langle\psi|e^{-it(x_j-x_i)\hat{\mathbf{B}}}|\psi\rangle\big|$ of the sum above decay to zero as $t\to\infty$ for all $i,j; i\neq j$; it can be shown to be a necessary condition by using the bound in Theorem 3.2.2. With the latter in mind, let us first introduce some elements from spectral theory in order to understand when QSD is asymptotically possible for the setting at hand.

3.5.1 Spectral Decomposition and Spectral Measures

Given some Self-Adjoint operator $\hat{\mathbf{A}}$ it is known that the residual spectrum of a self-adjoint operator is empty [8][3]. Hence, given that $\hat{\mathbf{A}}$ is self-adjoint we have

$$\mathbf{Spec}(\hat{\mathbf{A}}) = \mathbf{Spec}_p(\hat{\mathbf{A}}) \cup \mathbf{Spec}_{ac}(\hat{\mathbf{A}}) \cup \mathbf{Spec}_{sc}(\hat{\mathbf{A}}) \tag{3.145}$$

where the subscripts *ac* and *sc* stand for absolutely continuous and singular continuous respectively. In order to more formally define absolutely continuous and singular continuous spectra let us consider an arbitrary $|\psi\rangle \in \mathscr{H}$; $\mathscr{H}$ being the Hilbert space that $\hat{\mathbf{A}}$ acts in. The spectral theory then says that there exists a unique measure μ_ψ such that [3]

$$\langle\psi|\hat{\mathbf{A}}|\psi\rangle = \int_{\mathbb{R}}\lambda d\mu_\psi(\lambda). \tag{3.146}$$

The measure μ_ψ is often called the spectral measure generated by $|\psi\rangle$. By the *Lebesgue Decomposition Theorem* one may decompose any measure of this type into its point measure, absolutely continuous measure, and singular continuous measure components. i.e.

$$\mu_\psi = \mu_{\psi,p} + \mu_{\psi,ac} + \mu_{\psi,sc}. \tag{3.147}$$

Of particular interest to us will be the properties of the respective Fourier transforms of each of the measures on the right-hand side of (3.147). It is a consequence of the Riemann Lebegues Lemma that the Fourier transform of the measure $\mu_{\psi,ac}$ (absolutely continuous with respect to the Lebesgue measure) is a function that decays to zero as the argument becomes large. On the other hand, It can be shown that the Fourier transform of $\mu_{\psi,p}$ will exhibit quasiperiodic behavior while the Fourier transform of $\mu_{\psi,sc}$ (e.g.Cantor distribution (devils staircase), Dirac measure) is known not to decay to zero in general; However, there exist singular continuous measures with respective Fourier transform exhibiting the decay properties expected from the Fourier transforms of absolutely continuous measures. We shall be particularly interested in the subset of measures continuous with respect to the Lebesgue measures whose Fourier transform decays to zero. These will provide the necessary dynamics for the bound (3.2.3) to converge to zero for the case of the mixture presented above, i.e. $\sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_{i,t}$.

Let $\hat{\mathbf{A}}$ be a self-adjoint operator acting in some Hilbert space $\mathscr{H}$. The Hilbert space that $\hat{\mathbf{A}}$ acts in may furthermore be expressed as a direct sum of three invariant subspaces; one corresponding to each type of spectrum. Namely, from [8]

$$\mathscr{H} = \mathscr{H}_p \oplus \mathscr{H}_{ac} \oplus \mathscr{H}_{sc}. \tag{3.148}$$

Recall that the QSD problem may be solved asymptotically iff $\forall i \neq j$

$$\left|\left\langle\psi\middle|e^{-it(x_j-x_i)\hat{\mathbf{B}}}\middle|\psi\right\rangle\right| \to 0 \quad (as \quad t \to \infty) \tag{3.149}$$

Using the spectral theorem for unitary operators [3] it immediately follows that (3.149) may be written as

$$\left|\left\langle\psi\middle|e^{-it(x_j-x_i)\hat{\mathbf{B}}}\middle|\psi\right\rangle\right| = \left|\int_{\mathbb{R}} e^{-it(x_j-x_i)\lambda} d\mu_{\psi}(\lambda)\right| = \tag{3.150}$$

$$\left|\int_{\mathbb{R}} e^{-it(x_j-x_i)\lambda} d\mu_{\psi,p}(\lambda)\right| + \left|\int_{\mathbb{R}} e^{-it(x_j-x_i)\lambda} d\mu_{\psi,ac}(\lambda)\right| + \left|\int_{\mathbb{R}} e^{-it(x_j-x_i)\lambda} d\mu_{\psi,sc}(\lambda)\right|. \tag{3.151}$$

It is now clear why we are interested in the Fourier transforms of the p, ac and sc measures of the operator $\hat{\mathbf{B}}$. If we expect $\left|\left\langle\psi\middle|e^{-it(x_j-x_i)\hat{\mathbf{B}}}\middle|\psi\right\rangle\right| \to 0$ as $t \to \infty$, then we know that it will be necessary (but not sufficient!) for $\left|\psi\right\rangle \in \mathscr{H}_{ac} \oplus \mathscr{H}_{sc}$ [58]; first and foremost, $\mathscr{H}_{ac} \oplus \mathscr{H}_{sc}$ would of course have to be non-empty. However, not all $\left|\psi\right\rangle \in \mathscr{H}_{ac} \oplus \mathscr{H}_{sc}$ will yield the desired dynamics as we already discussed; everything in $\mathscr{H}_{ac}$ will yield the dynamics we want but not everything in $\mathscr{H}_{sc}$. We must hence constrain ourselves further the subspace $\mathscr{H}_{rc}$ consisting only of the states $\left|\psi\right\rangle$ whose associated measure μ_{ψ} is a *Rajchman* measure [58] (defined below). The associated invariant subspace is exactly what we need.

Definition 3.5.1 (Rajchman Measure:)

A finite Borel probability measure μ on $\mathbf{R}$ is called a Rajchman measure if it satisfies

$$\lim_{|t|\to\infty} \hat{\mu}(t) = 0 \tag{3.152}$$

where $\hat{\mu}(t) := \int_{\mathbb{R}} e^{2i\pi tx} d\mu(x) \quad , t \in \mathbb{R}$.

Theorem 3.5.1 (The Rajchman Subspace is a Closed Subspace)

Let $\hat{\mathbf{A}}$ be a self-adjoint operator acting on some arbitrary Hilbert space $\mathscr{H}$, then the set of vectors in $\mathscr{H}$ for which the spectral measure is a Rajchman measure, i.e.

$$\mathscr{H}_{rc} := \Big\{ \big|\psi\big\rangle \mid \lim_{|t|\to\infty} \big\langle\psi\big|e^{-it\hat{\mathbf{A}}}\big|\psi\big\rangle = 0 \Big\}, \tag{3.153}$$

is a closed subspace which is invariant under $e^{-is\hat{\mathbf{A}}}$[58].

Lemma 3.5.1 (If μ_ψ is Rajchman, then $\mu_{\phi,\psi}$ is Rajchman)

Let $\hat{\mathbf{B}}$ be some self-adjoint operator acting on a Hilbert space $\mathscr{H}$. Furthermore, let $\big|\psi\big\rangle \in \mathscr{H}_{rc}$ and $\big|\phi\big\rangle \in \mathscr{H}$, then the respective measure $\mu_{\phi,\psi}$ is Rajchman.

Proof.

$$\int e^{-it\lambda}\mu_{\phi,\psi}(\lambda) = \big\langle\phi\big|e^{-it\hat{\mathbf{B}}}\big|\psi\big\rangle = \big\langle\phi\big|\Big(\hat{\mathbf{P}}_{rc}e^{-it\hat{\mathbf{B}}}\big|\psi\big\rangle\Big) = \tag{3.154}$$

$$\Big(\big\langle\phi\big|\hat{\mathbf{P}}_{rc}\Big)e^{-it\lambda}\big|\psi\big\rangle = \big\langle\xi\big|e^{-it\hat{\mathbf{B}}}\big|\psi\big\rangle \tag{3.155}$$

where $\hat{\mathbf{P}}_{rc}$ is the projector onto the subspace $\mathscr{H}_{rc}$ and $\big|\xi\big\rangle := \hat{\mathbf{P}}_{rc}\big|\phi\big\rangle \in \mathscr{H}_{rc}$. We have used the fact that the Rajchman subspace is invariant under the action of $e^{-it\hat{\mathbf{B}}}$. Now, using the polarization identity (see [62] chapter 2 Excersise 2.1) we have

$$\big\langle\xi\big|e^{-it\hat{\mathbf{B}}}\big|\psi\big\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \Bigg(\Big(\big\langle\xi\big| + (-i)^k\big\langle\psi\big|\Big)e^{-it\hat{\mathbf{B}}}\Big(\big|\xi\big\rangle + i^k\big|\psi\big\rangle\Big)\Bigg) = \tag{3.156}$$

$$\frac{1}{4}\sum_{k=0}^{3} i^k \big\langle\chi_k\big|e^{-it\hat{\mathbf{B}}}\Big|\chi_k\big\rangle \tag{3.157}$$

where we have defined $\big|\chi_k\big\rangle : \big|\xi\big\rangle + i^k\big|\psi\big\rangle$. $\mathscr{H}_{rc}$ is a linear space, hence $\big|\chi_k\big\rangle \in \mathscr{H}_{rc}$ for $k = 0, 1, 2, 3$. Piecing all together.

$$\int e^{-it\lambda}\mu_{\phi,\psi}(\lambda) = \frac{1}{4}\sum_{k=0}^{3} i^k \big\langle\chi_k\big|e^{-it\hat{\mathbf{B}}}\Big|\chi_k\big\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \int e^{-it\lambda} d\mu_{\chi_k}(\lambda). \tag{3.158}$$

As $t \to \infty$ $\int e^{-it\lambda} d\mu_{\chi_k}(\lambda) \to 0$. Hence $\int e^{-it\lambda}\mu_{\phi,\psi}(\lambda) \to 0$ as $t \to \infty$. □

We conclude this subsection with the following proposition.

Proposition 3.5.1 (Full Solvability of QSD for URM of Pure States)

Consider the model described in this section by the states (3.67). $|\psi\rangle \in \mathscr{H}_{rc}$ corresponding to $\hat{\mathbf{B}}$ iff

$$\lim_{t\to\infty} \min_{POVM} p_E\Big\{\{p_i,\ e^{-itx_i\hat{\mathbf{B}}}|\psi\rangle\langle\psi|e^{itx_i\hat{\mathbf{B}}}\}_{i=1}^{N}, \{\hat{\mathbf{M}}_l\}_{l=1}^{K}\Big\} = 0 \tag{3.159}$$

Proof. This immediately follows from Theorems 3.2.3 and 3.2.2. □

3.6 Unitarily Related Mixtures of Finite Mixtures

Let us now consider the case where

$$\hat{\boldsymbol{\rho}} = \sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_i \in \mathcal{S}(\mathscr{H}), \tag{3.160}$$

with

$$\hat{\boldsymbol{\rho}}_i = \sum_{j=1}^{M} \eta_{ij} |\phi_{ij}\rangle\langle\phi_{ij}| \tag{3.161}$$

With all of the $|\phi_{ij}\rangle\langle\phi_{ij}| \in \mathcal{S}(\mathscr{H})$ and $\sum_j \eta_{ij} = 1$. In such a case we may again utilize Theorem 3.2.3 to begin with.

$$\min_{POVM} p_E(t) \leq \sum_i \sum_{j;j\neq i} \sqrt{p_i p_j} \sqrt{F\big(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j\big)} \tag{3.162}$$

however, the fidelities in this case are not immediately manageable owing to the fact that both $\hat{\boldsymbol{\rho}}_i$ and $\hat{\boldsymbol{\rho}}_j$ are booth mixed states. To overcome this hurdle we will use a bound for quantum fidelities found in [65]. Namely,

Theorem 3.6.1 (Fidelity Bound Koenraad and Milan [65])

Let $\sum_i p_i \hat{\boldsymbol{\rho}}_i$ be an arbitrary countable mixture and let $\hat{\boldsymbol{\sigma}}$ be an arbitrary density operator; both acting on the same arbitrary Hilbert space. Then,

$$\sqrt{F\Big(\sum_i p_i \hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\sigma}}\Big)} \leq \sum_i \sqrt{p_i} \sqrt{F\big(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\sigma}}\big)} \tag{3.163}$$

Note that this theorem for the general case of an infinite mixture would require that $\sqrt{p_i} \in \ell(\mathbb{R})$ in or more knowledge regarding the fidelities $\sqrt{p_i}\sqrt{F\big(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\sigma}}\big)}$ in order for us to profit from such a bound.

Applying Theorem 3.6.1 twice we may further bound (3.162) to obtain

$$\min_{POVM} p_E \leq \sum_i \sum_{j;j\neq i} \sqrt{p_i p_j} \sum_{k=1}^{M} \sum_{k'=1}^{M} \sqrt{\eta_{ik}} \sqrt{\eta_{ik'}} \sqrt{F\big(|\phi_{ik}\rangle\langle\phi_{ik}|, |\phi_{jk'}\rangle\langle\phi_{jk'}|\big)} = \tag{3.164}$$

$$\sum_i \sum_{j;j\neq i} \sqrt{p_i p_j} \sum_{k=1}^{M} \sum_{k'=1}^{M} \sqrt{\eta_{ik}\eta_{jk'}} \left|\left\langle \phi_{ik} \middle| \phi_{jk'} \right\rangle\right| \tag{3.165}$$

We hence see that the optimal probability error may be controlled by the inner products $\left|\left\langle \phi_{ik} \middle| \phi_{jk} \right\rangle\right|$ $(i \neq j)$, which are relatively easy to compute. We now provide a generalization to Proposition 3.5.1.

Theorem 3.6.2 (Full Solvability of QSD for URM of Finite Mixtures)

Let $\mathscr{H}$ be infinite-dimensional Hilbert space. Let $\hat{\mathbf{B}}$ be a self-adjoint operator acting in $\mathscr{H}$ with a non-empty Rajchman subspace. Furthermore, let $\hat{\boldsymbol{\rho}}_i := \sum_{j=1}^{M_i} \eta_{ij} \left|\phi_{ij}\right\rangle\left\langle\phi_{ij}\right|$ be finite mixtures in $\mathcal{S}\left(\mathscr{H}\right)$ for each i. Then,

$$\lim_{t\to\infty} \min_{POVM} p_E \left\{ \left\{ p_i,\ e^{-itx_i\hat{\mathbf{B}}} \hat{\boldsymbol{\rho}}_i e^{itx_i\hat{\mathbf{B}}} \right\}_{i=1}^{N}, \left\{\hat{\mathbf{M}}_l\right\}_{l=1}^{K} \right\} = 0 \tag{3.166}$$

iff all of the $\left|\phi_{ij}\right\rangle \in \mathscr{H}_{rc}$ of $\hat{\mathbf{B}}$.

Proof. First we assume that $\left|\phi_{ij}\right\rangle \in \mathscr{H}_{rc}$ of $\hat{\mathbf{B}}$ for all ij. Now, using (3.165) we have

$$\min_{POVM} p_E(t) \leq \sum_i \sum_{j;j\neq i} \sqrt{p_i p_j} \sum_{k=1}^{M_i} \sum_{k'=1}^{M_j} \sqrt{\eta_{ik}\eta_{jk'}} \left|\left\langle \phi_{ik} \middle| e^{-it(x_j - x_i)\hat{\mathbf{B}}} \middle| \phi_{jk} \right\rangle\right| \tag{3.167}$$

Since all of the sums above are finite, we need only worry about the limits

$$\lim_{t\to\infty} \left|\left\langle \phi_{ik} \middle| e^{-it(x_j - x_i)\hat{\mathbf{B}}} \middle| \phi_{jk} \right\rangle\right| \tag{3.168}$$

but by Lemma 3.5.1 these all go to zero as $t \to \infty$. We have therefore proven one direction of the theorem.

Going the other way we shall prove the contrapositive. Assume that $\left|\phi_{ij}\right\rangle \notin \mathscr{H}_{rc}$ for all ij. Using Theorem 3.2.2, followed by Theorem 3.2.5, we have

$$\min_{POVM} p_E(t) \geq \frac{1}{2} \sum_i \sum_{j;j\neq i} p_i p_j F\left(e^{-tx_i\hat{\mathbf{B}}} \hat{\boldsymbol{\rho}}_i e^{tx_i\hat{\mathbf{B}}}, e^{-tx_j\hat{\mathbf{B}}} \hat{\boldsymbol{\rho}}_{\boldsymbol{j}} e^{tx_i\hat{\mathbf{B}}} \right) \geq \tag{3.169}$$

$$\frac{1}{2} \sum_i \sum_{j;j\neq i} p_i p_j \left(\sum_{k=1}^{\min\left\{M_i, M_j\right\}} \sqrt{\eta_{ik}\eta_{jk}} \left|\left\langle \phi_{ik} \middle| e^{-it(x_j - x_i)\hat{\mathbf{B}}} \middle| \phi_{jk} \right\rangle\right|^2 \right)^2 \tag{3.170}$$

In this case the terms $\left|\left\langle \phi_{ik} \middle| e^{-it(x_j - x_i)\hat{\mathbf{B}}} \middle| \phi_{jk} \right\rangle\right|^2$ will be bounded away from zero infinitely often. Making $\min_{POVM} p_E(t)$ bounded away from zero infinitely often. Hence, asymptotic QSD is impossible. The theorem has been proved. □

Corollary 3.6.1 (QSD 2 with $\sum_j \sqrt{\eta_{ij}} < \infty$ for all j)

Theorem 3.6.2 may be extended to the cases where the finite mixtures $\hat{\boldsymbol{\rho}}_i$ are replaced by infinite mixtures $\hat{\boldsymbol{\rho}}_i := \sum_{j=1}^{\infty} \eta_{ij} |\phi_{ij}\rangle\langle\phi_{ij}|$, where now $\sum_{j=1}^{\infty} \eta_{ij} = 1$ for all i, if $\sum_j \sqrt{\eta_{ij}} < \infty$ for all i. The argument follows by applying the dominated convergence theorem to the first part of our proof for Theorem 3.6.2.

Corollary (3.6.1) gives us a way to work with the spectral decomposition of the operators in the mixtures $\sum_i p_i e^{-itx_i\hat{\mathbf{B}}} \hat{\boldsymbol{\rho}}_i e^{itx_i\hat{\mathbf{B}}}$, so long as the sequence $\sqrt{\lambda_{ij}}$ of square-rooted eigenvalues of each $\hat{\boldsymbol{\rho}}_i$ is summable with respect to j.

3.7 Uncountable Mixtures

Consider the case where instead of a countable mixture, as seen in (3.1), we have an uncountable one.

$$\hat{\boldsymbol{\rho}}_t := \int p(x) \hat{\boldsymbol{\rho}}_{x,t} dx \tag{3.171}$$

where $\hat{\boldsymbol{\rho}}_{x,t} := e^{-itx\hat{\mathbf{B}}} |\psi\rangle\langle\psi| e^{itx\hat{\mathbf{B}}}$, $|\psi\rangle\langle\psi|$ some initial state in $\mathcal{S}(\mathscr{H})$, $\mathscr{H}$ an infinite dimensional Hilbert space, $\hat{\mathbf{B}}$ a self-adjoint operator acting in $\mathscr{H}$ and $\int p(x) = 1$. The states $\hat{\boldsymbol{\rho}}_{x,t}$ are akin to the archetypal ensembles which are the main focus of QSD. In the literature [22] [25] [72] [27] for QSD, one almost always encounters ensembles of the form $\sum_i p_i \hat{\boldsymbol{\rho}}_i$ (p_i is a discrete probability distribution) and the task is of course to find a POVM that minimizes $\sum_i p_i Tr\{\hat{\boldsymbol{\rho}}_i - \hat{\boldsymbol{M}}_i \hat{\boldsymbol{\rho}}_i \hat{\boldsymbol{M}}_i^\dagger\}$ while satisfying $\sum_i \hat{\mathbf{M}}_i^\dagger \hat{\mathbf{M}}_i = \mathbb{I}$. If we wanted to discriminate all of the $\hat{\boldsymbol{\rho}}_x$ from each other, with high precision, we would expect that $F(\hat{\boldsymbol{\rho}}_x, \hat{\boldsymbol{\rho}}_y)$ should go to zero as $t \to \infty$ for all $x \neq y$. To see that the latter is not the case in general, recall that $F(\hat{\boldsymbol{\rho}}_x, \hat{\boldsymbol{\rho}}_y) = |\langle\psi| e^{-it(y-x)\hat{\mathbf{B}}} |\psi\rangle|$. Indeed, for fixed $x \neq y$, $|\langle\psi| e^{-it(y-x)\hat{\mathbf{B}}} |\psi\rangle| \to 0$ as $t \to \infty$ whenever $|\psi\rangle \in \mathscr{H}_{rc}$ (Rajchman subspace associated with $\hat{\mathbf{B}}$). However, if we choose y and x at every t such that $x - y = \frac{\alpha}{t}$, then it is clear that no matter how large t is there will always be x and y ($x \neq y$) such that $F(\hat{\boldsymbol{\rho}}_x, \hat{\boldsymbol{\rho}}_y) = |\langle\psi| e^{-i\alpha\hat{\mathbf{B}}} |\psi\rangle|$. If α is small, then $F(\hat{\boldsymbol{\rho}}_x, \hat{\boldsymbol{\rho}}_y)$ may be close to one. We therefore abandon the idea of discriminating all of the $\hat{\boldsymbol{\rho}}_x$ from one another and will instead support ourselves on the already existing theory of QSD for countable mixtures. We will do this by defining an N-mixture associated with the uncountable mixture (3.171). To motivate the latter let us first consider a partition of the support of $p(x)$ with N terms, i.e. $\cup_{i=1}^N \Omega_i = \mathbf{supp}\{p(x)\}$. Using this partition we may rewrite (3.171) as follows.

$$\int p(x) \hat{\boldsymbol{\rho}}_x dx = \sum_{i=1}^{N} \int_{\Omega_i} p(x) \hat{\boldsymbol{\rho}}_{x,t} dx. \tag{3.172}$$

Next we define some new terms, $p_i := \int_{\Omega_i} p(x) dx$ and $\bar{p}(x) := \frac{p(x)}{p_i}$. With these terms we further rewrite (3.172).i.e.

$$\sum_{i=1}^{N} \int_{\Omega_i} p(x) \hat{\boldsymbol{\rho}}_{x,t} dx = \sum_{i=1}^{N} p_i \Lambda_{i,t}(|\psi\rangle\langle\psi|) \tag{3.173}$$

Where $\Lambda_{i,t}\big(|\psi\rangle\langle\psi|\big) := \int_{\Omega_i} \bar{p}(x)e^{-itx\hat{\mathbf{B}}}|\psi\rangle\langle\psi|e^{itx\hat{\mathbf{B}}}dx := \int_{\Omega_i} \bar{p}(x)e^{-itx\hat{\mathbf{B}}}|\psi\rangle\langle\psi|e^{itx\hat{\mathbf{B}}}dx$. Notice that in (3.174) p_i is a discrete probability distribution and each $\Lambda_{i,t}\big(|\psi\rangle\langle\psi|\big)$ is indeed a density operator (not pure). The fact that the $t=0$ state was defined to be pure was immaterial, so we replaced it with a general state in the following. Let us now formally define an N-mixture of a particular uncountable mixture.

Definition 3.7.1 (N-mixture)

Let $\hat{\boldsymbol{\rho}} := \int p(x)\hat{\boldsymbol{\rho}}_x dx$ be some uncountable mixture. We call the following an N-mixutre of $\hat{\boldsymbol{\rho}}$ with respect to some partition $\cup_{i=1}^{N}\Omega_i$ (of N elements) of the support of $p(x)$.

$$\sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_{i,t} \tag{3.174}$$

where $p_i := \int_{\Omega_i} p(x)dx$, $\bar{p}(x) := \frac{p(x)}{p_i(t)}$ and $\hat{\boldsymbol{\rho}}_{i,t} := \int_{\Omega_i} \bar{p}(x)\hat{\boldsymbol{\rho}}_{x,t}dx$. We emphasize that this is not an approximation but merely a way of rewriting $\hat{\boldsymbol{\rho}}$; also note that the $\hat{\boldsymbol{\rho}}_i$ are density operators.

Given the mixture (3.174), we can use the theory of countable mixture QSD in order to estimate an optimal POVM that minimizes $\sum_i p_i Tr\big\{\hat{\boldsymbol{\rho}}_{i,t} - \hat{\boldsymbol{M}}_{i,t}\hat{\boldsymbol{\rho}}_{i,t}\hat{\boldsymbol{M}}_{i,t}^{\dagger}\big\}$ (in this case $\hat{\boldsymbol{\rho}}_{i,t} := \int_{\Omega_i} \bar{p}(x)\hat{\boldsymbol{\rho}}_{x,t}dx$), and in the case where finding the minimizing POVM is not possible we may bound the min error by making use of the Knill Barnum bound (3.29) [27] in order to study the theoretical effectiveness of the related QSD problem with respect to t, i.e. we would like to know if the associated QSD problem is *fully solvable* with respct to t or not. We now formalize the QSD problem for uncountable mixtures (UQSD).

Definition 3.7.2 (QSD for uncountable mixtures (UQSD))

Let $\mathscr{H}$ be an arbitrary Hilbert space. Now, consider the unaccountably mixed state $\hat{\boldsymbol{\rho}}_t := \int p(x)e^{-itx\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{itx\hat{\mathbf{B}}}dx$, $p(x)$ a probability density, $\hat{\boldsymbol{\rho}} \in \mathcal{S}\big(\mathscr{H}\big)$ some initial state, and $\hat{\mathbf{B}}$ a self adjoint operator (acting in $\mathscr{H}$). Furthermore, consider an N-mixture of $\hat{\boldsymbol{\rho}}_t$ with respect to some partition of the support of $p(x)$, $\cup_{i=1}^{N}\Omega_i$ (N elements). We call the associated optimization problem below the UQSD problem induced by the partition $\cup_{i=1}^{N}\Omega_i$,

$$\min_{POVM} \sum_{i=1}^{N} p_i\Big(1 - Tr\big\{\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_{i,t}\hat{\mathbf{M}}_i^{\dagger}\big\}\Big) \tag{3.175}$$

where now $p_i := \int_{\Omega_i} p(x)dx$, $\bar{p}_i(x) := \frac{p(x)}{p_i}$ and $\hat{\boldsymbol{\rho}}_{i,t} := \int_{\Omega_i} \bar{p}_i(x)e^{-itx\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{itx\hat{\mathbf{B}}}dx$.

An uncountable number of N-mixtures may be generated for any given uncountable mixture. If no constraints on the magnitudes of the Ω_i are posed, trivial N mixtures might be devised. e.g. consider the case of a 2-Mixture for some uncountable mixture $\int p(x)\hat{\boldsymbol{\rho}}_x dx$. For every $\varepsilon > 0$ we may choose Ω_2 such that $\big\| \int_{\Omega_2} \bar{p}(x)\hat{\boldsymbol{\rho}}_x dx\big\|_1 = \varepsilon$. Consequently $\big\| \int_{\Omega_1} \bar{p}(x)\hat{\boldsymbol{\rho}}_x dx\big\|_1 \geq 1-\varepsilon$ where $\Omega_1 = \mathbf{supp}\big\{p(x)\big\} - \Omega_2$.

Using the Hellström bound we get the following result.

$$\min_{POVM} p_E\Big\{\Big\{p_i, \int_{\Omega_i} p(x)\hat{\boldsymbol{\rho}}_x dx\Big\}_{i=1}^2, \{\hat{\mathbf{M}}_l\}_{l=1}^2\Big\} = \tag{3.176}$$

$$\frac{1}{2} - \frac{1}{2}\Big\| \int_{\Omega_1} p(x)\hat{\boldsymbol{\rho}}_x dx - \int_{\Omega_2} p(x)\hat{\boldsymbol{\rho}}_x dx\Big\|_1 \leq \tag{3.177}$$

$$\frac{1}{2} - \frac{1}{2}\Big|\Big\| \int_{\Omega_1} p(x)\hat{\boldsymbol{\rho}}_x dx\Big\|_1 - \Big\| \int_{\Omega_2} p(x)\hat{\boldsymbol{\rho}}_x dx\Big\|_1\Big| = \tag{3.178}$$

$$\frac{1}{2} - \frac{1}{2}\Big|\Big\| \int_{\Omega_1} p(x)\hat{\boldsymbol{\rho}}_x dx\Big\|_1 - \varepsilon\Big| \leq \frac{1}{2} - \frac{1}{2}\Big|1 - 2\varepsilon\Big| \approx 0 \tag{3.179}$$

We will hence only be interested in the UQSD optimization problem induced by partitions $\cup_{i=1}^N \Omega_i$ with magnitude constraints. i.e. $\Delta_{i,L} \leq |\Omega_i| \leq \Delta_{i,U}$ for every i. e.g. $|\Omega_i| > \Delta$ for all i. Which N-mixtures are physical and which are not is a question that has not been addressed as of yet and would be and be a problem best addressed through the lens of *Quantum Metrology* [15], a topic which is beyond the scope of this work.

Now that we have defined QSD for uncountable mixtures, we may ask ourselves if an adaptation of Proposition 3.5.1 is possible for this setting. Morally speaking this must be so; however, due to the intractability of the fidelity $F(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}})$ for the case where both $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\sigma}}$ are not pure states, the argument is not as direct as it was in Proposition 3.5.1 and owing to the uncountably mixed nature of the operators $\hat{\boldsymbol{\rho}}_{i,t}$ from (3.175) we may not apply the techniques all of the techniques used in proving Theorem 3.6.2. We encapsulate the latter in the latter discussion in the following conjecture.

Proposition 3.7.1 (Necessary and Sufficient Conditions for Full Solvability of UQSD for URM:)

Consider the setup of Definition 3.7.2. We conjecture that the UQSD optimization problem induced by a partition $\cup_{i=1}^N \Omega_i$ is fully solvable as $t \to \infty$ iff $\hat{\boldsymbol{\rho}} \in \mathcal{S}(\mathscr{H}_{rc})$, where $\mathscr{H}_{rc}$ is the Rajchman subspace of the operator $\hat{\mathbf{B}}$.

As a finishing note, we motivate Conjecture 3.7.1. For this, we will need the super fidelity.

Theorem 3.7.1 (Super Fidelity [71])

For any two density operators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\sigma}}$, then

$$F(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}) \leq Tr\{\hat{\boldsymbol{\rho}}\hat{\boldsymbol{\sigma}}\} + \sqrt{\big(1 - Tr\{\hat{\boldsymbol{\rho}}^2\}\big)\big(1 - Tr\{\hat{\boldsymbol{\sigma}}^2\}\big)} \tag{3.180}$$

Let us now consider the uncountable unitarily related mixture $\int p(x)e^{-itx\hat{\mathbf{B}}}|\psi\rangle\langle\psi|e^{itx\hat{\mathbf{B}}}dx$ where $|\psi\rangle \in \mathscr{H}_{rc}$ of $\hat{\mathbf{B}}$. Let $p(x)$ be a bimodal probability density $p(x) = \frac{1}{2}(p_1(x) + p_2(x))$ of two probability densities with non overlapping compact support. Let $\Delta_1 \subset \mathbb{R}$ and $\Delta_2 \subset \mathbb{R}$ be their supports respectively. Now, for Δ_1 and Δ_2 with any magnitude, i.e. $\delta_1 := |\Delta_1|$ and $\delta_2 := |\Delta_2|$ and for any $\varepsilon_1 > 0$

we may find a time domain $\mathscr{T} := [0, T_\varepsilon]$ so that

$$Tr\left\{\left(\int p_i(x)e^{-itx\hat{\mathbf{B}}}\big|\psi\big\rangle\big\langle\psi\big|e^{itx\hat{\mathbf{B}}}dx\right)^2\right\} \geq 1-\varepsilon \tag{3.181}$$

for all $t \in \mathscr{T}$ and $i = 1, 2$. Furthermore, with $\mathscr{T}$ fixed, for any $\varepsilon_2 > 0$ we can choose $\mathbf{dist}\big(\Delta_1, \Delta_2\big)$ such that

$$Tr\left\{\int p_1(x)e^{-itx\hat{\mathbf{B}}}\big|\psi\big\rangle\big\langle\psi\big|e^{itx\hat{\mathbf{B}}}dx \int p_2(x)e^{-itx\hat{\mathbf{B}}}\big|\psi\big\rangle\big\langle\psi\big|e^{itx\hat{\mathbf{B}}}dx\right\} < \varepsilon_2 \tag{3.182}$$

Proof. Fix $\varepsilon_2 > 0$ and let $t' \in \mathscr{T}$. Now,

$$Tr\left\{\int p_1(x)e^{-itx\hat{\mathbf{B}}}\big|\psi\big\rangle\big\langle\psi\big|e^{itx\hat{\mathbf{B}}}dx \int p_2(x)e^{-itx\hat{\mathbf{B}}}\big|\psi\big\rangle\big\langle\psi\big|e^{itx\hat{\mathbf{B}}}dx\right\} = \tag{3.183}$$

$$\int\int p_1(x)p_2(x)\Big|\big\langle\psi\big|e^{-it(y-x)\mathbf{B}}\big|\psi\big\rangle\Big|^2 dxdy \tag{3.184}$$

$\big|\psi\big\rangle \in \mathscr{H}_{rc}$ of $\hat{\mathbf{B}}$ hence $\lim_{\alpha\to\infty}\big\langle\psi\big|e^{-i\alpha\hat{\mathbf{B}}}\big|\psi\big\rangle = 0$. This means that there exists a $\delta > 0$ such that for all $\alpha > \delta$ we have $|\big\langle\psi\big|e^{-i\alpha\hat{\mathbf{B}}}\big|\psi\big\rangle| \leq \sqrt{\varepsilon_2}$. Choosing $\mathbf{dist}\big(\Delta_1.\Delta_2\big)t' > \delta$ we therefore have

$$\int\int p_1(x)p_2(x)\Big|\big\langle\psi\big|e^{-it(y-x)\mathbf{B}}\big|\psi\big\rangle\Big|^2 dxdy \leq \int\int p_1(x)p_2(x)|\sqrt{\varepsilon_2}|^2 dxdy = \varepsilon_2 \tag{3.185}$$

for all $t \in [t', T]$. □

With the latter and the use of Theorem 3.7.1 we have the following result.

$$F\left(\int p_1(x)e^{-itx\hat{\mathbf{B}}}\big|\psi\big\rangle\big\langle\psi\big|e^{itx\hat{\mathbf{B}}}dx,\ \int p_2(x)e^{-itx\hat{\mathbf{B}}}\big|\psi\big\rangle\big\langle\psi\big|e^{itx\hat{\mathbf{B}}}dx\right) \leq \varepsilon_2 + \varepsilon_1 \tag{3.186}$$

meaning that we may approximately solve the UQSD problem for the 2-mixture $\sum_{i=1}^{2} p_i\hat{\boldsymbol{\rho}}_i$ where $p_1 = p_2 = \frac{1}{2}$ and of course $\hat{\boldsymbol{\rho}}_i := \int p_i(x)e^{-itx\hat{\mathbf{B}}}\big|\psi\big\rangle\big\langle\psi\big|e^{itx\hat{\mathbf{B}}}dx$. We can proceed similarly for a multi-mode probability density. We can always place the lumps far enough apart from one order in such a way that we may observe decay in the optimal probability of error long before we see an error in our bound due to our super fidelity estimate estimates. It is clear that $\big|\psi\big\rangle \in \mathscr{H}_{rc}$ of $\hat{\mathbf{B}}$ plays a key role in going from (3.184 to (3.185); without this assumption our conclusion is not attainable.

3.7.1 UQSD Using a Particular PVM

We have already seen a successful $POVM$ scheme for the discretely mixed state QSD problem in Theorem 3.3.1. We can adapt such a setting to the case of an analogous uncountable mixture; we assume the same transnational dynamics for the states $\hat{\boldsymbol{\rho}}_{x,t} := e^{-itx\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{itx\hat{\mathbf{B}}}$, we may apply a similar $POVM$ as was applied in Theorem 3.3.1, but in this case the POVM will be a countably infinite set of projector operators, i.e. an N-mixture per Definition 3.7.1 (N may be infinity if the support of $p(x)$ is the entire real line). We will assume a specific partitioning of the support of $p(x)$ uniform in time. Namely $\Delta_i = ((\frac{x_{i-1}+x_i}{2}), (\frac{x_i+x_{i+1}}{2}))$ for $1 < i < N-1$, $\Delta_1 := \Big(\inf\big(\mathbf{supp}(p(x))\big),\ (\frac{x_0+x_1}{2})\Big)$, $\Delta_N := \Big((\frac{x_{N-1}+x_N}{2}),\ \sup\big(\mathbf{supp}(p(x))\big)\Big)$ and $\{x_i\}_{i=0}^{N-1}$ is a net of the support of $p(x)$. Furthermore let $\Omega_{i,t} := (t(\frac{x_{i-1}+x_i}{2}), t(\frac{x_i+x_{i+1}}{2}))$ for $1 < i < N-1$, $\Omega_{1,t} := \Big(\inf\big(t\mathbf{supp}(p(x))\big),\ t(\frac{x_0+x_1}{2})\Big)$, $\Omega_{N,t} := \Big(t(\frac{x_{N-1}+x_N}{2}),\ t\sup\big(\mathbf{supp}(p(x))\big)\Big)$. We will use the same N for all t To tackle the uncountable QSD problem, Definition 3.7.2, for this associated N-mixture we utilize the PVM $\{\hat{\mathbf{P}}_{\Omega_{i,t}}\}_i$, which are projectors projecting onto the subspaces generated by the subset $\Omega_{i,t}$. Using such a POVM the respective uncountable QSD problem is bounded as follows.

$$\min_{POVM}\sum_{i=1}^{N} p_i(t)\Big(1 - Tr\big\{\int_{\Delta_i} \bar{p}_i(t,x)\hat{\mathbf{M}}_{i,t}\hat{\boldsymbol{\rho}}_{x,t}\hat{\mathbf{M}}^{\dagger}_{i,t}dx\big\}\Big) \leq \tag{3.187}$$

$$1 - \sum_{i=1}^{N} p_i(t)\int_{\Delta_i}\bar{p}_i(x)\int_{t\frac{x_{i-1}+x_i}{2}-tx}^{t\frac{x_{i+1}+x_i}{2}-tx} K(x',x')dx'dx. \tag{3.188}$$

We see that as t becomes large (3.188) indeed approaches 0, meaning that there exists a POVM such that the QSD problem, Definition (3.7.2), is approximately solved with high accuracy. The latter follows from the fact that the limits of integration $t\frac{x_{i-1}+x_i}{2} - tx$ and $t\frac{x_{i+1}+x_i}{2} - tx$ are respectively negative and positive for all $x \in \Omega_{i,t}$ (except for the endpoints, which form a set of measure zero; causing no issues in the integration). The latter means that the sets $\Big[t\frac{x_{i-1}+x_i}{2} - tx,\ t\frac{x_{i+1}+x_i}{2} - tx\Big]$ approaches $\mathbb{R}$ as $t \to \infty$. Hence, $\int_{t\frac{x_{i-1}+x_i}{2}-tx}^{t\frac{x_{i+1}+x_i}{2}-tx} K(x',x')dx' \to 1$ as $t \to \infty$ which leads to (3.188) going to zero as $t \to \infty$. Although a very specialized case, this Uncountable QSD problem instills further confidence in Conjecture 3.7.1 and gives a schematic of how one might generate estimates for uncountable QSD in the cases where the Hilbert space in question is some space of square-integrable functions and the unitary dynamics in question are translational.

3.8 What if the $\hat{\mathbf{B}}$ is Finite-Dimensional

The previous discussions might lead one to believe that there is no asymptotic QSD for the case where $\hat{\mathscr{H}}$ is a finite-dimensional Hilbert paces. Indeed homologs to Theorem 3.6.2, Corollary 3.6.1, Propositions 3.5.1 and 3.7.1 is not possible per se, but one may approximately solve the associated QSD problem within some relevant time domain $t \in [0,T]$. To see this let us once again consider the inner products $\langle\psi_{i,t}|\psi_{j,t}\rangle$ that we scrutinized in the previous sections. In this case, however, $\hat{\mathbf{B}}$ will be finite-dimensional and have pure point spectrum (eigenvalues) b_i with associated eigenvectors $|b_i\rangle$.

$$\langle\psi_{i,t}|\psi_{j,t}\rangle = \langle\psi_0|e^{-it(x_j-x_i)\hat{\mathbf{B}}}|\psi_0\rangle = \sum_l e^{-it(x_j-x_i)b_l}|\langle b_j|\psi_0\rangle|^2. \tag{3.189}$$

A sum such as 3.189 is an *almost-periodic* function. There are various ways in which almost-periodic functions are defined, but we will stick with the definition posed by [57] (which interestingly enough, is a paper authored by Niels Bohr's brother Harold Bohr.).

Definition 3.8.1 (Uniformly almost-periodic functions)

A function is said to be uniformly almost-periodic if it lies in the closure of the trigonometric polynomials with respect to the uniform norm $\|f\|_\infty := \sup_x |f(x)|$

It is clear that (3.189) lies in the closure of the space of trigonometric polynomials since each of the terms in the sum is a periodic function of t which may be approximated to arbitrary precision via a series of complex exponentials with the same period. The sum (3.189) may therefore be estimated by a multi-indexed sum (one index for each term in (3.189)) which consists of trigonometric monomials. In [57] it was proven that Definition 3.8.1 is equivalent to the existence of relatively dense sets of so-called ε-periods, for all $\varepsilon > 0$; i.e. translations by $\tau(\varepsilon)$ (a time parameter dependent on ε) of the t resulting in the following bounds.

$$|f(t + \tau(\varepsilon)) - f(t)| \leq \varepsilon \tag{3.190}$$

where $f(x)$ is some almost-periodic function.

Our hurdles stemming from (3.189) may now be formally described. We would like to achieve an approximate asymptotic QSD in t for the case where $\hat{\mathbf{B}}$ acts in a finite-dimensional Hilbert space. For this to occur we will need the inner products 3.189 corresponding to each $\hat{\mathbf{B}}$ to be small. Being a finite-dimensional operator, $\hat{\mathbf{B}}$ does not have an associated Rajchman subspace. We must therefore rely on the largeness of the dimension of $\hat{\mathbf{B}}$ and the size of the particular time domain of interest. We have that the respective function (3.189) is almost-periodic. Hence, given some arbitrary time domain $t \in [0, T]$ there exists the possibility that

$$\Big| \sum_l e^{-i0(x_j - x_i)b_l} |\langle b_j|\psi_0\rangle|^2 - \sum_l e^{-i\tau(x_j - x_i)b_j} |\langle b_l|\psi_0\rangle|^2 \Big| = \tag{3.191}$$

$$\Big| \sum_l |\langle b_l|\psi_0\rangle|^2 - \sum_j e^{-i\tau(x_j - x_i)b_j} |\langle b_j|\psi_0\rangle|^2 \Big| = \Big| 1 - \sum_l e^{-i\tau(x_j - x_i)b_l} |\langle b_j|\psi_0\rangle|^2 \Big| < \varepsilon \tag{3.192}$$

for some arbitrary small $\varepsilon > 0$ and $0 \leq \tau \leq T$. i.e. it could be the case that our operator (4.40) returns to the state it was in at t= 0 after having evolved for an amount τ of the unit time. Since the time window of interest is $[0, T]$ and $\tau \leq T$, this would mean that we can not achieve SBS in such a time domain. This of course follows immediately from the bound in Theorem 3.2.2. In such a case the lower bound of Theorem 3.2.2 is maximized.

If however, we are given some time interval $[0, T]$, where $\langle \psi_{i,t} | \psi_{j,t} \rangle$ is sharply decaying with respect to some time scale τ much smaller than T(the relevant time frame) and $\langle \psi_{i,t} | \psi_{j,t} \rangle$ stabilizes to zero long before t approaches T, then we may conclude that convergence Asymptotic QSD with respect to the time scale τ is achievable. To see that the latter may be achieved let us consider the case where $|\psi_0\rangle$ is some tensor product state

$$|\psi_0\rangle = \bigotimes_{k=1}^{N_{mac}} |\psi_0^k\rangle \tag{3.193}$$

and

$$\hat{\mathbf{B}} = \sum_{k=1}^{N_{mac}} \hat{\mathbf{B}}_k \tag{3.194}$$

with each $\left|\psi_0^k\right\rangle \in \mathscr{H}_k$ (finite dimensional Hilbert spaces) and $\hat{\mathbf{B}}_k$ some linear self-adjoint operator acting in $\mathscr{H}_k$. In this case

$$\left\langle\psi_{i,t}\middle|\psi_{j,t}\right\rangle = \left(\bigotimes_{k=1}^{N_{mac}} \left\langle\psi_0^k\right|\right)\left(e^{-it(x_j-x_i)\sum_{k=1}^{N_{mac}} g_k\hat{\mathbf{B}}_k}\right)\left(\bigotimes_{k=1}^{N_{mac}} \left|\psi_0^k\right\rangle\right) = \tag{3.195}$$

$$\left(\bigotimes_{k=1}^{N_{mac}} \left\langle\psi_0^k\right|\right)\left(\bigotimes_{k=1}^{N_{mac}} e^{-it(x_j-x_i)g_k\hat{\mathbf{B}}_k}\left|\psi_0^k\right\rangle\right) = \prod_{k=1}^{N_{mac}} \left\langle\psi_0^k\middle|e^{-itg_k(x_j-x_i)\hat{\mathbf{B}}_k}\middle|\psi_0^k\right\rangle \tag{3.196}$$

$$\prod_{k=1}^{N_{mac}} \sum_j e^{-itg_k(x_j-x_i)b_j^k}\left|\left\langle b_j^k\middle|\psi_0^k\right\rangle\right|^2. \tag{3.197}$$

Each term in the latter product consists of a bounded sum

$$\Big|\sum_l e^{-itg_k(x_j-x_i)b_l^k}\left|\left\langle b_j^k\middle|\psi_0^k\right\rangle\right|^2\Big| \leq 1. \tag{3.198}$$

The magnitude of (3.197) is therefore bounded between 0 and 1, since it is a product of terms whose respective magnitudes are also bounded between 0 and 1. If N_{mac} is large enough, then for a given $\tau \in [0,T]$ it can be shown that $\left|\left\langle\psi_{i,t}\middle|\psi_{j,t}\right\rangle\right|$ may be bounded from above via a time-independent bound of desired magnitude. We now formalize the latter. Let $0 < \varepsilon' < 1$ and assume that for all k, $1 - \left|\sum_l e^{-itg_k(x_j-x_i)b_l^k}\right| > \varepsilon'_k > 0$ for all $t \in [\tau, T]$ ($\tau > 0$ and $\tau << T$), then the functions $\sum_l e^{-itg_k(x_j-x_i)b_l^k}\left|\left\langle b_j^k\middle|\psi_0^k\right\rangle\right|^2$ do not have ε-periods for $\varepsilon = \varepsilon'_k$. i.e.

$$\left|1 - \sum_l e^{-itg_k(x_j-x_i)b_l^k}\left|\left\langle b_j\middle|\psi_0\right\rangle\right|^2\right| \geq \left|1 - \left|\sum_l e^{-itg_k(x_j-x_i)b_l^k}\left|\left\langle b_j\middle|\psi_0\right\rangle\right|^2\right|\right| = \tag{3.199}$$

$$1 - \left|\sum_l e^{-itg_k(x_j-x_i)b_l^k}\left|\left\langle b_j\middle|\psi_0\right\rangle\right|^2\right| \geq \varepsilon'_k. \tag{3.200}$$

Furthermore,

$$\Big|\sum_l e^{-itg_k(x_j-x_i)b_l^k}\left|\left\langle b_j\middle|\psi_0\right\rangle\right|^2\Big| \leq 1 - \varepsilon'_k \tag{3.201}$$

for all $t \in [\tau, T]$. Note that $1 - \varepsilon'_k < 1$, hence picking up from (3.197) we have

$$\left|\prod_{k=1}^{N_{mac}} \sum_j e^{-itg_k(x_j-x_i)b_j^k}\left|\left\langle b_j^k\middle|\psi_0^k\right\rangle\right|^2\right| \leq \prod_{k=1}^{N_{mac}} (1-\varepsilon'_k) \leq (1 - \min_k \varepsilon'_k)^{N_{mac}}. \tag{3.202}$$

We may therefore conclude that if we desire

$$\left|\prod_{k=1}^{N_{mac}} \sum_j e^{-it(x_j-x_i)b_j^k}\left|\left\langle b_j^k\middle|\psi_0^k\right\rangle\right|^2\right| \leq \prod_{k=1}^{N_{mac}} (1-\varepsilon'_k) = \delta \tag{3.203}$$

for some small $\delta > 0$, we can choose $N_{mac} = \frac{\ln(\delta)}{\ln(1-\min_k \varepsilon'_k)}$ in order to obtain such a bound. We, therefore, see that the size of the parameter N_{mac} is responsible for the smallness of (3.197). Returning now to the question of asymptotic QSD for finite-dimensional Hilbert spaces; we have seen that such a QSD problem may be approximately solved over a time domain $[\tau, T]$ in a sense, given a large enough N_{mac}. Very quickly after τ time elapses (3.197) attains the smallness desired and therefore we achieve an approximately discriminable state for the time interval $[\tau, T]$ with $\tau << T$.

Chapter 4

SBS for Discrete Variables

4.1 Work by Jarek et all

In recent times significant attention has been given to a family of multipartite states named *Spectrum Broadcast Structures* (SBS) [38] [39] [40] [53]. Since its genesis, the theory of SBS has been used as a tool in the discipline of *Quantum Foundations*; particularly in the theories of *Quantum Decoherence* and *Quantum Darwinism*[12][31][41][42]. Recently, quantum darwinism and SBS theory have been shown to be equivalent under certain technical assumptions[43]. Motivating the theory of quantum darwinism and the theory of SBS is the question of objectivity in the quantum world. To avoid philosophical contention [38] [39] and[40] provide a definition of objectivity motivated by properties of classical dynamical systems. The multipartite quantum mechanical state satisfying such properties is called a SBS. The definition of objectivity proposed in [38] is:

Definition 4.1.1 (Objectivity [38][39][40])

A state of the system S exists objectively if many observers can find out the state of S independently, and without perturbing it.

There are two clauses in the definition above that are ambiguous, namely, "can find out the state of S" and "without perturbing it". The first of these means that any of the observers may locally solve a QSD (3.21) estimation problem that allows the observer to identify the state of the system S by proxy. The second clause, "without perturbing it" may be formalized by introducing a distance measure. We will only be using the trace distance, but different distance measures may be more relevant in other scenarios. The following definition proposed in [38] is a mathematical formalization of Definition 4.1.1 and is what we will refer to as a SBS.

Definition 4.1.2 (SBS [38][39][40])

A Spectrum Broadcast Structure is a multipartite state (also called joint state) of a system S and an environment E, consisting of sub-environments $E^1, E^2, ..., E^{N_E}$:

$$\hat{\boldsymbol{\rho}}_{SBS} = \sum_i p_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_i^{E^k} \tag{4.1}$$

where $\{|i\rangle\}_i$ is some basis in the system's space, p_i are probabilities summing to one, and all states $\hat{\boldsymbol{\rho}}_i^{E^k}$ are perfectly distinguishable in the following sense:

$$F\left(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k}\right) = 0 \tag{4.2}$$

for all $i \neq j$ and for all $k = 1, ..., N_E$. Recall that $F(..., ...)$ is the quantum fidelity, and is defined as $F\left(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}\right) := \left\|\sqrt{\hat{\boldsymbol{\rho}}}\sqrt{\hat{\boldsymbol{\sigma}}}\right\|_1^2$ (2.44).

In [38] it is argued that SBS satisfies the desired definition of objectivity and that it is the only structure that satisfies such a definition. The argument for why the observer monitoring (i.e. employing measurements characterized via a POVM) E^l may find out the state of S independently is the following. Let us analyze the local state pertaining to E^l; to do this we partially trace out the degrees of freedom pertaining to the system S and all of the environments E^k with the exception of the lth environment. i.e. from (4.1) we obtain

$$\sum_i p_i \left(\prod_{k \neq l} Tr_{E^k}\left\{\hat{\boldsymbol{\rho}}_i^{E^k}\right\} \right) \langle i|i\rangle \hat{\boldsymbol{\rho}}_i^{E^l} = \sum_i p_i \hat{\boldsymbol{\rho}}_i^{E^l}. \tag{4.3}$$

Notice that this is a mixed state. If $F\left(\hat{\boldsymbol{\rho}}_i^{E^l}, \hat{\boldsymbol{\rho}}_j^{E^l}\right) = 0$ for $i \neq j$ then the QSD problem may be optimally solved. This means that there exists a POVM which the observer monitoring the environment E^l may utilize to conduct measurements on E^l yielding perfect distinguishability between the possible outcomes of the mixture (4.3). Furthermore, the state $\hat{\boldsymbol{\rho}}_i^{E^l}$ is correlated with the state $|i\rangle\langle i|$ of S in the sense that when S is found to be in the state $|i\rangle\langle i|$ the lth environment will be found in the state $\hat{\boldsymbol{\rho}}_i^{E^l}$. Owing to the perfect distinguishability between the states $\hat{\boldsymbol{\rho}}_i^{E^l}$ for all i, there is no ambiguity regarding the state of S given that E^l is found to be in the state $\hat{\boldsymbol{\rho}}_i^{E^l}$. Since l was taken to be arbitrary it is clear that any environmental observer may find out the state of S faithfully so long as $F\left(\hat{\boldsymbol{\rho}}_i^{E^l}, \hat{\boldsymbol{\rho}}_j^{E^l}\right) = 0$ for $i \neq j$ is satisfied.

To argue non-disturbance (a similar approach follows for approximate non-disturbance) we first re-emphasize that the "can find out" in Definition 4.1.1 formally means that for every E^k there exists a POVM $\{\hat{\mathbf{E}}_i^{E^k}\}_i$ that solves the respective local QSD problem, i.e. that discriminates perfectly the mixture (4.3). $\{\bigotimes_{k=1}^{N_E} \hat{\mathbf{E}}_{i_k}^{E^k}\}_{i_1, i_2, ..., i_{N_E}}$ will hence be a POVM acting on $\mathcal{S}\left(\mathscr{H}_S \bigotimes_{k=1}^{N_E} \mathscr{H}_{E^k}\right)$. If the POVM optimally solving the local QSD problem for each environment E^l does so in a non-perturbing way, i.e. not changing the state after the associated measurement quantum channel has been applied in the trace distance sense, then the measurement associated with the POVM $\{\bigotimes_{k=1}^{N_E}\hat{\mathbf{E}}_{i_k}^{E^k}\}_{i_1,i_2,\ldots,i_{N_E}}$ may be shown to also be non-disturbing with respect to the trace distance. i.e. it can be shown that

$$\frac{1}{2}\left\|\sum_i p_i|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_i^{E^k}-\sum_{i_1}\sum_{i_2}\ldots\sum_{i_{N_E}}\left(\sum_i p_i|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{M}}_{i_k}^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_{i_k}^{E^k}\right)^\dagger\right)\right\|_1=0 \tag{4.4}$$

where $\left(\hat{\mathbf{M}}_{i_k}^{E^k}\right)^\dagger\hat{\mathbf{M}}_{i_k}^{E^k}=\hat{\mathbf{E}}_{i_k}^{E^k}$. i.e. Given the perfect distinguishability of the $\hat{\boldsymbol{\rho}}_i^{E^k}$ for each k, a POVM $\{\hat{\mathbf{E}}_{i_k}^{E^k}\}_{i_k}$ may be devised such that

$$\hat{\mathbf{M}}_{i_k}^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_{i_k}^{E^k}\right)^\dagger=\delta_{i_k i}. \tag{4.5}$$

With (4.5) in mind, we may estimate the trace distance in (4.4).

$$\frac{1}{2}\left\|\sum_i p_i|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_i^{E^k}-\sum_{i_1}\sum_{i_2}\ldots\sum_{i_{N_E}}\left(\sum_i p_i|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{M}}_{i_k}^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_{i_k}^{E^k}\right)^\dagger\right)\right\|_1= \tag{4.6}$$

$$\frac{1}{2}\left\|\sum_i p_i|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_i^{E^k}-\sum_i p_i|i\rangle\langle i|\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\right\|_1\leq \tag{4.7}$$

$$\frac{1}{2}\sum_i p_i\left\|\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_i^{E^k}-\bigotimes_{k=1}^{N_E}\hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\right\|_1 \tag{4.8}$$

To proceed we introduce the following lemma.

Lemma 4.1.1 (Telescopic inequality [40])

Let $\hat{\mathbf{A}}^k$ and $\hat{\mathbf{B}}^k$ be trace class operators for all k. Then,

$$\left\|\bigotimes_{k=1}^{N}\hat{\mathbf{A}}^k-\bigotimes_{k=1}^{N}\hat{\mathbf{B}}^k\right\|_1\leq \tag{4.9}$$

$$\sum_{j=1}^{N}\left(\prod_{k=1}^{j-1}\left\|\hat{\mathbf{A}}^k\right\|_1\right)\times\left\|\hat{\mathbf{A}}^j-\hat{\mathbf{B}}^j\right\|_1\times\left(\prod_{k=j+1}^{N}\left\|\hat{\mathbf{B}}^k\right\|_1\right) \tag{4.10}$$

Using Lemma 4.1, (4.8) may be bounded as follows.

$$\frac{1}{2}\sum_i p_i\left\|\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}_i^{E^k}-\bigotimes_{k=1}^{N_E}\hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\right\|_1\leq\frac{1}{2}\sum_{k=1}^{N_E}\sum_i p_i\left\|\hat{\boldsymbol{\rho}}_i^{E^k}-\hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\right\|_1. \tag{4.11}$$

We claim that the distinguishability criterion $F\left(\hat{\boldsymbol{\rho}}_i^{E^k},\hat{\boldsymbol{\rho}}_j^{E^k}\right)=0$ ($i\neq j$) for all k is a necessary and sufficient condition for (4.11) to vanish. For the case of perfect distinguishability, the sufficiency is immediately clear since each $\hat{\mathbf{E}}_i^{E^k}$ may be chosen to be a projector onto the domain of $\hat{\boldsymbol{\rho}}_i^{E^k}$ respectively, meaning that $\hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger=\hat{\boldsymbol{\rho}}_i^{E^k}$ which in turn implies that $\|\hat{\boldsymbol{\rho}}_i^{E^k}-\hat{\mathbf{M}}_i^{E^k}\hat{\boldsymbol{\rho}}_i^{E^k}\left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\|_1=0$.

The argument becomes more transparent in the case where all of the $\hat{\boldsymbol{\rho}}_i^{E^k}$ are projectors. In such a case we simply choose $\hat{\mathbf{E}}_i^{E^k} = \hat{\boldsymbol{\rho}}_i^{E^k}$.

The distinguishability condition $F\big(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k}\big) = 0$ $(i \neq j)$ for all k is of course an idealization; in practice there will always be some error involved in the distinguishability measures $F\big(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k}\big) = \varepsilon_k$ for all k where $\varepsilon_k > 0$ will depend on dynamical parameters such as time. In such a case we must tread more carefully. In the previous paragraph, we did not need to calculate or estimate the trace norm present because we showed that the operator in the trace norm was the zero operator. If the perfect distinguishability condition is not satisfied, then we will need to compute/estimate the sum over i of trace norms in (4.11). Although we may use Theorem 3.2.3 in order to bound $\min_{POVM} \sum_i p_i Tr\{\hat{\boldsymbol{\rho}}_i^{E^k} - \hat{\mathbf{M}}_i^{E^k} \hat{\boldsymbol{\rho}}_i^{E^k} \left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\}$ by fidelitites of the set of density matrices $\{\hat{\boldsymbol{\rho}}_i^{E^k}\}_i$, i.e.

$$\min_{POVM} Tr\Big\{\hat{\boldsymbol{\rho}}_i^{E^k} - \hat{\mathbf{M}}_i^{E^k} \hat{\boldsymbol{\rho}}_i^{E^k} \left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\Big\} \leq \sum_i \sum_{j;j\neq i} \sqrt{p_i p_j} \sqrt{F(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_i)} \tag{4.12}$$

for all k, we do not yet have tools that aid us in the estimation of $\sum_i p_i \Big\| \hat{\boldsymbol{\rho}}_i^{E^k} - \hat{\mathbf{M}}_i^{E^k} \hat{\boldsymbol{\rho}}_i^{E^k} \left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger \Big\|_1$. It is clear by (4.12) that if the fidelities $F\big(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k}\big)$ $(i \neq j)$ are arbitrarily small, then for all k the local QSD error $\min_{POVM} Tr\Big\{\hat{\boldsymbol{\rho}}_i^{E^k} - \hat{\mathbf{M}}_i^{E^k} \hat{\boldsymbol{\rho}}_i^{E^k} \left(\hat{\mathbf{M}}_i^{E^k}\right)^\dagger\Big\}$ will also become arbitrarily small. To show that a similar argument holds for the right-hand side of the inequality (4.11) we will prove in the following section a bound for (4.8) that will depend only on fidelities between the set of density operators $\{\hat{\boldsymbol{\rho}}_i^{E^k}\}_i$, and vanish as the fidelities $F\big(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k}\big)$ $(i \neq j)$ decay to zero for all k.

4.2 Bounding the Super Quantum State Discrimination Problem (SQSD)

In this section, we shall be simplifying our notational conventions since we shall not need the superscripts on the density operators used in the previous section. Consider the mixed state $\sum_{i=1}^N p_i \hat{\boldsymbol{\rho}}_i$, where $\sum_{i=1}^N p_i = 1$ and the $\hat{\boldsymbol{\rho}}_i$ are pure states in a Hilbert space of dimension greater than N, i.e. one-dimensional projections $|\psi_i\rangle\langle\psi_i|$, where $\{|\psi_i\rangle\}_{i=1}^N$ are normalized vectors. Assuming that $|\psi_i\rangle$ are linearly independent, we may use the well-known Gram-Schmidt procedure to define the associated orthonormal set.

Theorem 4.2.1 (Gram-Schmidt Procedure)

Assume that the set $\{|\psi\rangle_i\}_{i=1}^N$, of vectors in some vector space V, is a linearly independent set. Then the following construction yields an orthonormal set.

$$|\phi_1\rangle = |\psi_1\rangle \tag{4.13}$$

$$|\phi_2\rangle = \frac{1}{\alpha_2}\Big(|\psi_2\rangle - \langle\phi_1|\psi_2\rangle|\phi_1\rangle\Big) \tag{4.14}$$

$$\vdots$$

$$|\phi_N\rangle = \frac{1}{\alpha_N}\Big(|\psi_N\rangle - \sum_{k=1}^{N-1}\langle\phi_k|\psi_N\rangle|\phi_k\rangle\Big) \tag{4.15}$$

Here $\alpha_i := \Big\||\psi_i\rangle - \sum_{k=1}^{i-1}\langle\phi_k|\psi_i\rangle|\phi_k\rangle\Big\| = \sqrt{1 - \sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle|^2}$ for $i > 1$ and $\alpha_1 = 1$ are the respective normalization constants. We have $Span\Big\{\{|\psi_i\rangle\}_{i=1}^N\Big\} = Span\Big\{\{|\phi_i\rangle\}_{i=1}^N\Big\}$.

The orthonormal set $\{|\phi_i\rangle\}_{i=1}^N$ may be used for the construction of a PVM, namely

$$\Big\{|\phi_i\rangle\langle\phi_i|\Big\}_{i=1}^N \cup \Big\{\mathbb{I} - \sum_{i=1}^N |\phi_i\rangle\langle\phi_i|\Big\} \tag{4.16}$$

which we will use to estimate $\min_{POVM}\sum_{i=1}^N p_i\|\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\|_1$:

$$\min_{POVM}\sum_{i=1}^N p_i Tr\Big\{\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\Big\} \le \min_{POVM}\sum_{i=1}^N p_i\|\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\|_1 \le \tag{4.17}$$

$$\sum_{i=1}^N p_i\Big\|\hat{\boldsymbol{\rho}}_i - |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\Big\|_1 \tag{4.18}$$

for a judiciously chosen PVM $\{|\phi_i\rangle\langle\phi_i|\}_i$.

Lemma 4.2.1 (Trace Distance Lemma)

Let $\hat{\boldsymbol{\rho}}_i$ and $|\phi_i\rangle$ be defined as above; also let $i > 1$, then

$$\|\hat{\boldsymbol{\rho}}_i - |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\|_1 \le 2\sum_{k=1}^{i-1}|\langle\phi_k|\psi_i\rangle| \tag{4.19}$$

Proof.

$$\Big\|\hat{\boldsymbol{\rho}}_i - |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\Big\|_1 = \Big\||\psi_i\rangle\langle\psi_i|\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\Big\|_1 = \tag{4.20}$$

$$\Big\|\Big(|\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big)\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i| + |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\Big(|\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big)\Big\|_1 \le \tag{4.21}$$

$$\left\|\Big(|\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big)\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i|\right\|_1 + \left\||\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\Big(|\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\Big)\right\|_1 \leq \tag{4.22}$$

$$\left\||\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\right\|_1 \left\|\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i|\right\|_1 + \left\||\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\right\|_1 \left\||\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\right\|_1 = \tag{4.23}$$

$$\left\||\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\right\|_1 \left(\left\|\hat{\boldsymbol{\rho}}_i|\psi_i\rangle\langle\psi_i|\right\|_1 + \left\||\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i\right\|_1\right) \leq \tag{4.24}$$

$$\left\||\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\right\|_1 \left(\|\hat{\boldsymbol{\rho}}_i\|_1 \left\||\psi_i\rangle\langle\psi_i|\right\|_1 + \left\||\phi_i\rangle\langle\phi_i|\right\|_1 \|\hat{\boldsymbol{\rho}}_i\|_1\right) \leq \tag{4.25}$$

$$2\left\||\psi_i\rangle\langle\psi_i| - |\phi_i\rangle\langle\phi_i|\right\|_1 = 2\sqrt{1 - |\langle\psi_i|\phi_i\rangle|^2} = \tag{4.26}$$

$$2\sqrt{1 - \left|\frac{1}{\alpha_i}\left(1 - \sum_{k=1}^{i-1} |\langle\phi_k|\psi_i\rangle|^2\right)\right|^2} = 2\sqrt{1 - \left|\frac{\left(1 - \sum_{k=1}^{i-1} |\langle\phi_k|\psi_i\rangle|^2\right)}{\sqrt{\left(1 - \sum_{k=1}^{i-1} |\langle\phi_k|\psi_i\rangle|^2\right)}}\right|^2} \tag{4.27}$$

$$= 2\sqrt{1 - 1 + \sum_{k=1}^{i-1} |\langle\phi_k|\psi_i\rangle|^2} = 2\sqrt{\sum_{k=1}^{i-1} |\langle\phi_k|\psi_i\rangle|^2} \leq 2\sum_{k=1}^{i-1} |\langle\phi_k|\psi_i\rangle| \tag{4.28}$$

□

The term $\sum_{k=1}^{i-1} |\langle\phi_k|\psi_i\rangle|$ may be understood in terms of the related *Gram Determinant*. We present this as a lemma.

Lemma 4.2.2 (Gramm Determinants)

$$|\phi_j\rangle = \frac{1}{\sqrt{D_{j-1}D_j}} \mathbf{det} \begin{pmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_j\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_j\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{j-1}|\psi_1\rangle & \langle\psi_{j-1}|\psi_2\rangle & \dots & \langle\psi_{j-1}|\psi_j\rangle \\ |\psi_1\rangle & |\psi_2\rangle & \dots & |\psi_j\rangle \end{pmatrix} \tag{4.29}$$

where

$$D_j := \mathbf{det} \begin{pmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_j\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_j\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_j|\psi_1\rangle & \langle\psi_j|\psi_2\rangle & \dots & \langle\psi_j|\psi_j\rangle \end{pmatrix} \tag{4.30}$$

where $D_0 := 1$.

In determinant form, $\langle\psi_i|\phi_k\rangle$ may now be written as follows.

$$\langle\psi_i|\phi_k\rangle = \frac{1}{\sqrt{D_{k-1}D_k}}\mathbf{det}\begin{pmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_k\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{k-1}|\psi_1\rangle & \langle\psi_{k-1}|\psi_2\rangle & \dots & \langle\psi_{k-1}|\psi_k\rangle \\ \langle\psi_i|\psi_1\rangle & \langle\psi_i|\psi_2\rangle & \dots & \langle\psi_i|\psi_k\rangle \end{pmatrix} \tag{4.31}$$

The power of viewing the states $|\phi_i\rangle$ in their determinant form is that now we need only compute inner products between elements of the set $\{|\psi_i\rangle\}_{i=1}^N$ in order to estimate the solution of the PVM (4.16) via an approximate solution to (4.8) problem with a particular PVM, i.e. $\min_{PVM}\sum_{i=1}^N p_i\left\|\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\right\|_1$. Recall that the states $\{|\psi_i\rangle\}_{i=1}^N$ are normalized and let us consider the case where $\langle\psi_i|\psi_j\rangle = \varepsilon_{ij}$ for all $i \neq j \in \{1, ..., N\}$, where ε_{ij} are complex numbers satisfying $|\varepsilon_{ij}| \leq \delta$ for all $i \neq j \in \{1, .., N\}$, where δ is small. Since, under this assumption, all entries of the last column of the matrix (4.31) are small, this would also imply that $\left\|\hat{\boldsymbol{\rho}}_i - |\phi_i\rangle\langle\phi_i|\hat{\boldsymbol{\rho}}_i|\phi_i\rangle\langle\phi_i|\right\|_1$ is small for all i, thanks to Lemma 4.2.1.

The above estimates imply the following theorem.

Theorem 4.2.2 (Bound for the Super Quantum State Discrimination Optimization Problem)

Consider a mixed state of the form $\sum_{i=1}^N p_i\hat{\boldsymbol{\rho}}_i$, $\sum_{i=1}^N p_i = 1$, where $\hat{\boldsymbol{\rho}}_i := |\psi_i\rangle\langle\psi_i|$ are pure states acting on a *Hilbert* space of dimension greater than N. Furthermore, assume that the states $\{|\psi_i\rangle\}_i$ are linearly independent. Then

$$\min_{POVM}\sum_{i=1}^N p_i\left\|\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\right\|_1 \leq \sum_{i=2}^N p_i\sum_{k=1}^{i-1}\left|\frac{M_{k,i}}{D_{k-1}D_k}\right| \tag{4.32}$$

where

$$M_{k,i} := 2\mathbf{det}\begin{pmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_k\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_{k-1}|\psi_1\rangle & \langle\psi_{k-1}|\psi_2\rangle & \dots & \langle\psi_{k-1}|\psi_k\rangle \\ \langle\psi_i|\psi_1\rangle & \langle\psi_i|\psi_2\rangle & \dots & \langle\psi_i|\psi_k\rangle \end{pmatrix} \tag{4.33}$$

$$D_k := \mathbf{det}\begin{pmatrix} \langle\psi_1|\psi_1\rangle & \langle\psi_1|\psi_2\rangle & \dots & \langle\psi_1|\psi_k\rangle \\ \langle\psi_2|\psi_1\rangle & \langle\psi_2|\psi_2\rangle & \dots & \langle\psi_2|\psi_k\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\psi_k|\psi_1\rangle & \langle\psi_k|\psi_2\rangle & \dots & \langle\psi_k|\psi_k\rangle \end{pmatrix} \tag{4.34}$$

Proof. The proof follows directly from Lemma 4.2.2 and Lemma 4.2.1, and the fact that for $i = 1$ the corresponding projector is simply $|\psi_i\rangle\langle\psi_i|$ making the $i = 1$ term zero. □

It is with the bound provided by Theorem 4.2.2 that we may estimate the right-hand side of (4.11). Notice that the magnitudes of the elements of the determinants found in (4.32) and (4.34) are all bounded by the square root of the respective fidelities. i.e noting that [9]

$$\langle \psi_i | \psi_j \rangle \leq \big| \langle \psi_i | \psi_j \rangle \big| = \sqrt{F\Big(|\psi_i\rangle\langle\psi_i|, |\psi_j\rangle\langle\psi_j| \Big)} \tag{4.35}$$

With the latter relationship, it is clear that the bound of Theorem 4.2.2 will consist purely of fidelities as was alluded to in the previous section.

4.3 Dynamical Monitoring for Discrete Variables

We have been studying properties of SBS states as introduced in Definition 4.1.2. In the present section we will focus on the convergence of a time-dependent density operator $\hat{\boldsymbol{\rho}}_t$ to an SBS state under quantum-mechanical time evolution. If we knew *a priori* that a certain type of multipartite quantum-mechanical system behaves objectively per Definitions 4.1.1 and 4.1.2, then the states of this system should converge to an SBS state as $t \to \infty$. Time evolution may in general be described by an arbitrary time-dependent quantum map. We will focus on the quantum maps obtained by partial trace from unitary evolution corresponding to a Hamiltonian specified below.

4.3.1 Quantum-Measurement Limit

The principal models studied in SBS literature [38][39][40] are of the *quantum-measurement* limit type, meaning SBS that arise from dynamics generated by Hamiltonians in which the interaction term between the system S and the environment E dominates, i.e. $\hat{\mathbf{H}}_{tot} = \hat{\mathbf{H}}_{int} + \hat{\mathbf{H}}_E + \hat{\mathbf{H}}_S \approx \hat{\mathbf{H}}_{int}$ (here *tot* means total and *int* means interaction terms). Such an approximation is valid when the system and the environments evolve with respect to a time scale that is much larger than that of the time scale corresponding to that of the interactive dynamics; these type of dynamics are central to the theory of *quantum decoherence* [31] . In this work, we will furthermore narrow our focus to interaction Hamiltonians of the von Neaumann type [18]

$$\hat{\mathbf{H}}_{int} = \hat{\mathbf{X}} \otimes \sum_{k=1}^{N} g_k \hat{\mathbf{B}}_k \tag{4.36}$$

The corresponding time evolution operator is hence

$$\hat{\mathbf{U}}_t = e^{-it\hat{\mathbf{X}} \otimes \sum_{k=1}^{N} g_k \hat{\mathbf{B}}_k}. \tag{4.37}$$

The theory of SBS for discrete variables focuses on the case where the system S is described by a finite-dimensional Hilbert space [38][39][40]. In this case, the self-adjoint operator $\hat{\mathbf{X}}$ has purely discrete spectrum. Let $\{|i\rangle\}_{i=1}^{d_S}$ be the set of eigenvectors of $\hat{\mathbf{X}}$ with x_i being the corresponding eigenvalues. $\hat{\mathbf{B}}_k$ is assumed to be an arbitrary self-adjoint operator. We shall see that the spectral properties of the operator $\hat{\mathbf{B}}$ will determine whether or not the multipartite states converge to an SBS state.

4.3.2 Partial Tracing

We consider a quantum system interacting with N macroscopic environments. We assume that the joint initial state has the product form:

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=1}^{N} \hat{\boldsymbol{\rho}}^{E_0^k} \tag{4.38}$$

In the state (4.38) we write the subscript 0 in E_0^k in order to emphasize that this is the initial state of the kth environment E^k; similarly we use the subscript in S_0 to highlight the initial state of the system. We evolve our total initial state using the evolution operator (4.37).

$$\hat{\boldsymbol{\rho}}_t = \left(e^{-it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N} g_k\hat{\mathbf{B}}_k}\right)\hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=1}^{N} \hat{\boldsymbol{\rho}}^{E_0^k}\left(e^{it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N} g_k\hat{\mathbf{B}}_k}\right). \tag{4.39}$$

To study the state of the subsystem formed by the system S and the first N_E environments, we take the partial trace of the time-evolved density operator over the remaining $M_E := N - N_E$ environments. The result is,

$$\sum_{i,j=1}^{d_S} \sigma_{i,j}\Gamma(i,j,t)|i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i,x_j} \tag{4.40}$$

where, again, $\{|i\rangle\}_{i=1}^{d_s}$ are the eigenvectors of $\hat{\mathbf{X}}$, with corresponding eigenvalues $\{x_i\}_{i=1}^{d_S}$ and we have use the following notation

$$\hat{\boldsymbol{\rho}}^{E_t^k}_{x,y} := e^{-itxg_k\hat{\mathbf{B}}_k}\hat{\boldsymbol{\rho}}^{E_0^k}e^{ityg_k\hat{\mathbf{B}}_k}\ (k = 1, 2, ..., N_E) \tag{4.41}$$

$$\hat{\boldsymbol{\rho}}^{E_t^k}_{x} := e^{-itxg_k\hat{\mathbf{B}}_k}\hat{\boldsymbol{\rho}}^{E_0^k}e^{itxg_k\hat{\mathbf{B}}_k}\ (k = 1, 2, ..., N_E). \tag{4.42}$$

$$\sigma_{i,j} := \langle i|\hat{\boldsymbol{\rho}}_{S_0}|j\rangle \tag{4.43}$$

$$\gamma^k_{i,j}(t) := Tr\{\hat{\boldsymbol{\rho}}^{E_t^k}_{x_i,x_j}\} \tag{4.44}$$

$$\Gamma(i,j,,t) := \prod_{n=N_E+1}^{N} \gamma^n_{i,j}(t) \tag{4.45}$$

We may also write (4.40) as

$$\Lambda_t\left(\hat{\boldsymbol{\rho}}_{S_0} \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_0^k}\right) := \sum_{i,j=1}^{d_S} \sigma_{i,j}\Gamma(i,j,t)|i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_t^k}_{x_i,x_j} \tag{4.46}$$

where Λ_t is a quantum channel (see [9] for a discussion on quantum channels) defined as follows.

$$\Lambda_t(\hat{\boldsymbol{\rho}}) := \mathscr{U}_t \circ \mathscr{E}_t(\hat{\boldsymbol{\rho}}) \tag{4.47}$$

where

$$\mathscr{U}_t(\hat{\mathbf{A}}) := e^{-it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N_E} g_k\hat{\mathbf{B}}_k}(\hat{\mathbf{A}})e^{it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N_E} g_k\hat{\mathbf{B}}_k} \tag{4.48}$$

acts non-trivially in $\mathcal{S}\big(\mathscr{H}_S \otimes \bigotimes_{k=1}^{N_E} \mathscr{H}_k\big)$ and $\mathscr{E}_t$ acts non-trivially only in $\mathcal{S}\big(\mathscr{H}_S\big)$ as follows.

$$\mathscr{E}_t(\hat{\mathbf{C}}) := \sum_{i,j=1}^{d_S} \langle i|\hat{\mathbf{C}}|j\rangle\Gamma(i,j,t)|i\rangle\langle j| \tag{4.49}$$

In words, the trace-preserving quantum map Λ_t is a composition of two trace-preserving quantum maps $\mathscr{U}_t$ and $\mathscr{E}_t$: a unitary map acting on S and the environmental degrees of freedom that were not traced out and a non-unitary map acting locally in S.

4.4 Monitoring the Process of System Information Broadcasting

In [40] the goal was to show that (4.40) converges to an associated SBS state as t goes to ∞. For a given $t \geq 0$, one can create an SBS state approximating (4.40) in the following way. We first restrict the sum of (4.40) to the diagonal terms—the terms with $i = j$. We will label the resulting operator as follows.

$$\hat{\boldsymbol{\rho}}_{dg,t} = \sum_{i=1}^{d_S} \sigma_i|i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \tag{4.50}$$

The next step is to choose for every t a PVM acting on the space $\mathcal{S}\big(\mathscr{H}_S \otimes \bigotimes_{k=1}^{N_E} \mathscr{H}_{E^k}\big)$ (Note that for the case considered in [40], $dim(\mathscr{H}_S) = d_S < \infty$ and $dim(\mathscr{H}_{E^k}) = d_{E^k} < \infty$ for all k). To define such a PVM, the eigenbasis of the operator $\hat{\mathbf{X}}$ is used: the elements of the PVM are of the form $\big|i\big\rangle\big\langle i\big| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_j^{E_t^k}$ where the $\big\{\big|i\big\rangle\big\langle i\big|\big\}_{i=1}^{d_S}$ and $\big\{\hat{\mathbf{P}}_j^{E_t^k}\big\}_{j=1}^{d_S} \cup \big\{\mathbb{I} - \sum_{i=1}^{d_S} \hat{\mathbf{P}}_i^{E_t^k}\big\}$ resolve the identity operators in $\mathcal{B}\big(\mathscr{H}_S\big)$ and $\mathcal{B}\big(\mathscr{H}_{E_k}\big)$ respectively, so that, in particular,$\big\{\hat{\mathbf{P}}_j^{E_t^k}\big\}_{j=1}^{d_S} \cup \big\{\mathbb{I} - \sum_{i=1}^{d_S} \hat{\mathbf{P}}_i^{E_t^k}\big\}$ is a PVM in the kth environment's Hilbert space. The latter PVMs are then used to approximate the operator (4.40) by an SBS state:

$$\hat{\boldsymbol{\rho}}_{SBS,t} := \frac{1}{\mathscr{N}}\sum_{j=1}^{d_S}\left(\big|j\big\rangle\big\langle j\big| \otimes \bigotimes_{k=1}^{N_E} \boldsymbol{P}_j^{E_t^k}\right)\hat{\boldsymbol{\rho}}_{diag,t}\left(\big|j\big\rangle\big\langle j\big| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_j^{E_t^k}\right) = \tag{4.51}$$

$$\sum_{i=1}^{d_S} \tilde{\sigma}_i|i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E}\left(\hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right). \tag{4.52}$$

Here $\mathscr{N}$ is a normalizing constant and $\tilde{\sigma}_i := \frac{\sigma_i}{\mathscr{N}}$. One can verify that the operator (4.52) is indeed an SBS state as defined in Definition 4.1.2. If (4.40) converges to an operator of the form (4.52) as $t \to \infty$, we say that (4.40) is asymptotically SBS. Convergence is meant here in the sense of trace distance. Namely, one would like to show that

$$\min_{PVM} \big\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t}\big\|_1 \to 0 \text{ as } t \to \infty \tag{4.53}$$

where for each t the minimization is taken over all projector-valued-measures $\{\hat{\mathbf{P}}_i^{E_t^k}\}_{i=1}^{d_S} \cup \{\mathbb{I} - \sum_{i=1}^{d_S} \hat{\mathbf{P}}_i^{E_t^k}\}$. Utilizing the fact that $\min_{POVM} \left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t}\right\|_1 \leq \min_{PVM} \left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t}\right\|_1$ we may conclude that (4.53) implies that $\min_{POVM} \left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t}\right\|_1 \to 0$ as $t \to \infty$ as well. An attempt is made in [40] to prove (4.53) but the argument provided there is incomplete. In what follows we discuss the bounds presented in [40], as well as propose and prove an alternative bound for the trace distance in (4.53).

In [40], the following bound is conjectured for the trace distance in (4.53).

$$\frac{1}{2} \min_{PVM} \left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t}\right\|_1 \leq \Gamma(t) + \sum_i \sum_{j;j\neq i} \sqrt{\sigma_i \sigma_j} \sum_{k=1}^{N_E} F\left(\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}, \hat{\boldsymbol{\rho}}_{x_j}^{E_t^k}\right) \tag{4.54}$$

where now, $\Gamma(t) := \sum_i \sum_{j;j\neq i} |\sigma_{i,j}| \prod_{k=N_E+1}^{N} |\gamma_{i,j}^k(t)|$, and again $\gamma_{i,j}^k(t) = Tr\left[\hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^k}\right]$, $\sigma_{i,j} := \langle i|\hat{\boldsymbol{\rho}}_{S_0}|j\rangle$. This result would allow to estimate the minimum on the LHS, using the asymptotic properties of $\Gamma(t)$ and the fidelity terms in (4.54). This bound would in turn provide a way to estimate $\frac{1}{2}\min_{POVM} \left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t}\right\|_1$. As (4.54) is currently not known to be true, we will not be using it. Instead, we will be utilizing the bound constituting Theorem 4.2.2 proven in the previous section.

4.4.1 A New Bound for the Trace Distance of a Multipartite State and an Approximating SBS State

In what follows we use an unnormalized version of (4.51): $\hat{\boldsymbol{\rho}}_{PSBS,t} := \mathscr{N}\hat{\boldsymbol{\rho}}_{SBS,t}$. In practice it is easier to bound $\left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{PSBS,t}\right\|_1$ and then use Lemma 4.4.1, stated below, to bound $\left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t}\right\|_1$.

Lemma 4.4.1 (Trace Distance Lemma)

For density operators $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\sigma}}$ $\left\|\hat{\boldsymbol{\rho}} - \eta\hat{\boldsymbol{\sigma}}\right\|_1 \leq L$ implies $\left\|\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}\right\| \leq 2L$ for constants $L \geq 0$ and $\eta \in [0, 1]$

Proof. Using reverse triangle inequality we see that

$$L \geq \|\hat{\boldsymbol{\rho}} - \eta\hat{\boldsymbol{\sigma}}\|_1 \geq \big|\|\hat{\boldsymbol{\rho}}\|_1 - \|\eta\hat{\boldsymbol{\sigma}}\|_1\big| = \|\hat{\boldsymbol{\rho}}\|_1 - \|\eta\hat{\boldsymbol{\sigma}}\|_1 = 1 - \eta \tag{4.55}$$

furthermore

$$\|\hat{\boldsymbol{\rho}} - \hat{\boldsymbol{\sigma}}\|_1 = \|\hat{\boldsymbol{\rho}} - \eta\hat{\boldsymbol{\sigma}} + \eta\hat{\boldsymbol{\sigma}} - \hat{\sigma}\|_1 \leq \|\hat{\boldsymbol{\rho}} - \eta\hat{\boldsymbol{\sigma}}\|_1 + \|\eta\hat{\boldsymbol{\sigma}} - \hat{\boldsymbol{\sigma}}\|_1 \leq \tag{4.56}$$

$$L + (1-\eta)\|\hat{\sigma}\|_1 = L + (1-\eta) \leq L + L = 2L \tag{4.57}$$

□

We now prove a preliminary inequality.

$$\left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{PSBS,t}\right\|_1 = \tag{4.58}$$

$$\left\| \sum_{i,j=1}^{d_S} \sigma_{i,j}\Gamma(i,j,t)|i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^k} - \sum_{i=1}^{d_S} \sigma_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 \leq \tag{4.59}$$

$$\left\| \sum_{i=1}^{d_S} \sigma_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \sum_{i=1}^{d_S} \sigma_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 + \left\| \sum_i \sum_{j;j\neq i}^{d_S} \sigma_{i,j}\Gamma(i,j,t)|i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^k} \right\|_1 \leq \tag{4.60}$$

$$\sum_{i=1}^{d_S} \left\| \sigma_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \sigma_i |i\rangle\langle i| \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 + \left\| \sum_i \sum_{j;j\neq i}^{d_S} \sigma_{i,j}\Gamma(i,j,t)|i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^k} \right\|_1 \leq \tag{4.61}$$

$$\sum_{i=1}^{d_S} \sigma_i \left\| |i\rangle\langle i| \otimes \left(\bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right) \right\|_1 + \sum_i \sum_{j;j\neq i}^{d_S} \left|\sigma_{i,j}\Gamma(i,j,t)\right| \left\| |i\rangle\langle j| \otimes \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^k} \right\|_1 = \tag{4.62}$$

$$\sum_{i=1}^{d_S} \sigma_i \left\| \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 + \sum_i \sum_{j;j\neq i}^{d_S} \left|\sigma_{i,j}\Gamma(i,j,t)\right| \leq \tag{4.63}$$

$$\sum_{k=1}^{N_E} \sum_{i=1}^{d_S} \sigma_i \left\| \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 + \sum_i \sum_{j;j\neq i}^{d_S} \left|\sigma_{i,j}\Gamma(i,j,t)\right| \tag{4.64}$$

where in the last step we have used Lemma 4.1. Now, using Lemma 4.4.1 we conclude that

$$\frac{1}{2} \min_{PVM} \left\| \hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBS,t} \right\|_1 \leq \min_{PVM} \left(\sum_{k=1}^{N_E} \sum_{i=1}^{d_S} \sigma_i \left\| \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 \right) + \Gamma(t) \tag{4.65}$$

$\Gamma(t) := \sum_i \sum_{j;j\neq i} |\sigma_{i,j}| \prod_{k=N_E+1}^{N} |\gamma_{i,j}^k(t)|$, $\gamma_{i,j}^k(t) = Tr\big[\hat{\boldsymbol{\rho}}_{x_i,x_j}^{E_t^k}\big]$, and $\sigma_{i,j} := \langle i|\hat{\boldsymbol{\rho}}_{S_0}|j\rangle$. In (4.65), $\Gamma(t)$ is the decoherence term which is independent of the choice of the PVM minimized over. The decoherence term is simple to study provided that we are able to compute the traces defining the terms $\gamma_{i,j}^k(t)$. The first term in (4.65) involves a minimization over all PVM for each value of t. Rather than attempting to solve the minimization problem exactly, we shall be employing Theorem 4.2.2 to bound (4.65).

In order to do this, you must assume that the initial states $\hat{\boldsymbol{\rho}}^{E_0^k}$ are pure; we will consider the case where these are not pure in Section 4.7. The purity of $\hat{\boldsymbol{\rho}}^{E_0^k}$ implies that the operators $\hat{\boldsymbol{\rho}}_i^{E_t^k}$ (using the notation defined in (4.42))are pure as well for all i since the evolution (4.42) is unitary. We will henceforth write $\hat{\boldsymbol{\rho}}^{E_0^k}$ as a projector.

$$\left|\psi_{i,t}^k\right\rangle\!\left\langle\psi_{i,t}^k\right| = \hat{\boldsymbol{\rho}}_i^{E_t^k} \tag{4.66}$$

We now use Theorem 4.2.2 to estimate the first summand of (4.65), therefore obtaining the following theorem.

Theorem 4.4.1 (Estimating Proximity to SBS)

Using the definitions found in this section so far,

$$\frac{1}{2}\min_{POVM}\left\|\hat{\boldsymbol{\rho}}_t-\hat{\boldsymbol{\rho}}_{SBS,t}\right\|_1\leq\frac{1}{2}\sum_{k=1}^{N_E}\sum_{i=2}^{d_S}\sigma_i\sum_{s=1}^{i-1}\left|\frac{M^k_{s,i}}{D^k_{s-1,t}D^k_{s,t}}\right|+\Gamma(t) \tag{4.67}$$

where

$$M^k_{s,i}:=\mathbf{det}\begin{pmatrix}\langle\psi^k_{1,t}|\psi^k_{1,t}\rangle & \langle\psi^k_{1,t}|\psi^k_{2,t}\rangle & \cdots & \langle\psi^k_{1,t}|\psi^k_{s,t}\rangle\\ \langle\psi^k_{2,t}|\psi^k_{1,t}\rangle & \langle\psi^k_{2,t}|\psi^k_{2,t}\rangle & \cdots & \langle\psi^k_{2,t}|\psi^k_{s,t}\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\psi^k_{s-1,t}|\psi^k_{1,t}\rangle & \langle\psi^k_{s-1,t}|\psi^k_{2,t}\rangle & \cdots & \langle\psi^k_{s-1,t}|\psi^k_{s,t}\rangle\\ \langle\psi^k_{i,t}|\psi^k_{1,t}\rangle & \langle\psi^k_{i,t}|\psi^k_{2,t}\rangle & \cdots & \langle\psi^k_{i,t}|\psi^k_{s,t}\rangle\end{pmatrix} \tag{4.68}$$

$$D^k_{s,t}:=\mathbf{det}\begin{pmatrix}\langle\psi^k_{1,t}|\psi^k_{1,t}\rangle & \langle\psi^k_{1,t}|\psi^k_{2,t}\rangle & \cdots & \langle\psi^k_{1,t}|\psi^k_{s,t}\rangle\\ \langle\psi^k_{2,t}|\psi^k_{1,t}\rangle & \langle\psi^k_{2,t}|\psi^k_{2,t}\rangle & \cdots & \langle\psi^k_{2,t}|\psi^k_{s,t}\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\psi^k_{j,t}|\psi^k_{1,t}\rangle & \langle\psi^k_{j,t}|\psi^k_{2,t}\rangle & \cdots & \langle\psi^k_{s,t}|\psi^k_{s,t}\rangle\end{pmatrix} \tag{4.69}$$

Given that computing determinants is a difficult task, one might wonder if there is a way to avoid doing so via further bounding the term (4.67) with another term that does not involve determinants. It turns out that such an approach is possible and if the entries $\langle\psi^k_{s,t}|\psi^k_{l,t}\rangle$ are small enough, the process is even easier to handle. We will develop such an approach in Section 4.5 of this chapter. The bound (4.67) may be used to dynamically monitor the convergence of some multipartite quantum state undergoing non-unitary evolution to an SBS state. Regardless of whether (4.54) is viable or not; the bound (4.67) gives us a way to monitor convergence to SBS in a way that is independent of the optimization problem over all PVMS and/or POVMS.

It is with Theorem 4.4.1 that we hope to mitigate the gap in [40]. Although Corollary 1 of [40] is not substantiated by a correct proof at the moment, we present Theorem 4.4.1 as a viable alternative to Corollary 1 of [40]. If fate should have it that Corollary 1 is shown to be fundamentally untrue, then Theorem 4.4.1 would be the only tool for us to choose from (i.e. to the extent of the author's knowledge).

4.5 Further Bounds for Theorem 4.2.2

As mentioned in the previous section, taking determinants is in general computationally costly. If one could find an estimate that allowed us to avoid computing the determinants of Theorem 4.4.1 this would be of great utility. We are typically interested in asymptotic behavior; in particular, we are studying cases where the minimization terms of Theorem 4.4.1 are expected to become small with respect to the relevant time scale of the dynamics in question (usually the decoherence time-scale is used). So long as we find a bound that shows the same asymptotic dynamics as the upper bound of

Theorem 4.4.1, such a bound may also allow us to estimate the smallness of the minimization terms in (Theorem 4.4.1) as t gets large with respect to the relevant time scale. To begin our search for such an estimate we introduce three results that we shall be using.

Theorem 4.5.1 (Hadamard's inequality for determinants [16])

Let $\hat{\mathbf{A}}$ be some arbitrary $N \times N$ matrix with entries $A_{i,j}$. Then

$$\mathbf{det}\big(\hat{\mathbf{A}}\big) \leq \prod_{j=1}^{N} \left(\sum_{i=1}^{N} |A_{ij}|^2 \right)^{\frac{1}{2}}.$$

Theorem 4.5.2

[16] Let $\mathbb{I} + \hat{\mathbf{B}}$ be an $N \times N$ matrix with entries $\delta_{ij} + B_{ij}$ where $B_{i,i} = 0$ for all i. Then

$$\mathbf{det}(\mathbb{I} + \hat{\mathbf{B}}) = \prod_{j=1}^{N} \big(1 + \lambda_j\big(\hat{B}\big)\big)$$

Theorem 4.5.3 (Gerschgorin Theorem [54])

Let $\hat{\mathbf{A}}$ be an arbitrary $N \times N$ matrix with matrix elements $A_{i,j}$. Now, define

$$\mathscr{D}_i := \left\{ z \in \mathbb{C} : |z - A_{ii}| \leq \sum_{j;j\neq i} |A_{ij}| \right\}.$$

Then, all of the eigenvalues of the operator $\hat{A}$ are found in the set $\mathscr{G}_N := \bigcup_{i=1}^{N} \mathscr{D}_i$. The sets $\mathscr{D}_i$ are known as Gerschgorin discs.

We now use these theorems to prove the following. Using the notational conventions of the previous section,

Theorem 4.5.4

$$\min_{PVM}\sum_{i=1}^{d_S}\sigma_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}-\mathbf{P}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\mathbf{P}_i^{E_t^k}\right\|_1\leq \tag{4.70}$$

$$d_S\Big(1+d_S M_{d_S}(t)\Big)^{d_S-1}\sum_{i\neq j}^{d_S}\frac{\sigma_i|\langle\psi_{i,t}^k|\psi_{j,t}^k\rangle|}{\min_{x\in\mathscr{G}_{d_S}}\big|1-|x|\big|^{i-1}} \tag{4.71}$$

where

$$\mathscr{G}_s:=\bigcup_{i=1}^{s}\mathscr{D}_i^s \tag{4.72}$$

$$\mathscr{D}_i^s:=\Big\{x\in\mathbb{R}:|x|\leq\sum_{j;j\neq i}|B_{ij,t}^s|\Big\}\ i\in\{1,...,k\} \tag{4.73}$$

$$M_{d_S}(t):=\max_{i\neq j;\{1,...,d_S\}}|\langle\psi_{i,t}^k|\psi_{j,t}^k\rangle| \tag{4.74}$$

and

$$\hat{\mathbf{B}}_t^s:=\begin{pmatrix}0&\langle\psi_{1,t}^k|\psi_{2,t}^k\rangle&\dots&\langle\psi_{1,t}^k|\psi_{s,t}^k\rangle\\ \langle\psi_{2,t}^k|\psi_{1,t}^k\rangle&0&\dots&\langle\psi_{2,t}^k|\psi_{s,t}^k\rangle\\ \vdots&\vdots&\ddots&\vdots\\ \langle\psi_{j,t}^k|\psi_{1,t}^k\rangle&\langle\psi_{j,t}^k|\psi_{2,t}^k\rangle&\dots&0\end{pmatrix} \tag{4.75}$$

Proof. Assume that $s>2$. Then, using Theorem 4.5.1

$$\hat{\mathbf{A}}:=\left|\mathbf{det}\begin{pmatrix}\langle\psi_{1,t}^k|\psi_{1,t}^k\rangle&\langle\psi_{1,t}^k|\psi_{2,t}^k\rangle&\dots&\langle\psi_{1,t}^k|\psi_{s,t}^k\rangle\\ \langle\psi_{2,t}^k|\psi_{1,t}^k\rangle&\langle\psi_{2,t}^k|\psi_{2,t}^k\rangle&\dots&\langle\psi_{2,t}^k|\psi_{s,t}^k\rangle\\ \vdots&\vdots&\ddots&\vdots\\ \langle\psi_{s-1,t}^k|\psi_{1,t}^k\rangle&\langle\psi_{s-1,t}^k|\psi_{2,t}^k\rangle&\dots&\langle\psi_{s-1,t}^k|\psi_{s,t}^k\rangle\\ \langle\psi_{i,t}^k|\psi_{1,t}^k\rangle&\langle\psi_{i,t}^k|\psi_{2,t}^k\rangle&\dots&\langle\psi_{i,t}^k|\psi_{s,t}^k\rangle\end{pmatrix}\right|\leq \tag{4.76}$$

$$\prod_{n=1}^{s}\Big(\sum_{m=1}^{s}|A_{n,m}|^2\Big)^{\frac{1}{2}} \tag{4.77}$$

Where

$$A_{nm}=\langle\psi_{n,t}^k|\psi_{m,t}^k\rangle\ \ for\ \ n\in\{1,...,k-1\}\ \ m\in\{1,...,k\} \tag{4.78}$$

and

$$A_{nm}=\langle\psi_{i,t}^k|\psi_{m,t}^k\rangle\ \ for\ \ n=k\ \ m\in\{1,...,k\}. \tag{4.79}$$

Therefore,

$$\prod_{n=1}^{s}\Big(\sum_{m=1}^{s}|A_{nm}|^2\Big)^{\frac{1}{2}}=\prod_{n=1}^{s-1}\Big(\sum_{m=1}^{s}|\langle\psi_{n,t}^k|\psi_{m,t}^k\rangle|^2\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{s}|\langle\psi_{i,t}^k|\psi_{m,t}^k\rangle|^2\Big)^{\frac{1}{2}}\leq \tag{4.80}$$

$$\left(\max_{n\in\{1,...,s-1\}}\sum_{m=1}^{s}|\langle\psi^k_{n,t}|\psi^k_{m,t}\rangle|^2\right)^{\frac{s-1}{2}}\left(\sum_{m=1}^{s}|\langle\psi^k_{i,t}|\psi^k_{m,t}\rangle|^2\right)^{\frac{1}{2}} \tag{4.81}$$

$$\left(1+\max_{n\in\{1,...,s-1\}}\sum_{m=1;m\neq n}^{s}|\langle\psi^k_{n,t}|\psi^k_{m,t}\rangle|^2\right)^{\frac{s-1}{2}}\left(\sum_{m=1}^{s}|\langle\psi^k_{i,t}|\psi^k_{m,t}\rangle|^2\right)^{\frac{1}{2}}\leq \tag{4.82}$$

$$\left(1+\max_{n\in\{1,...,d_S-1\}}\sum_{m=1;m\neq n}^{d_S}|\langle\psi^k_{n,t}|\psi^k_{m,t}\rangle|^2\right)^{\frac{d_S-1}{2}}\left(\sum_{m=1}^{s}|\langle\psi^k_{i,t}|\psi^k_{m,t}\rangle|^2\right)^{\frac{1}{2}}\leq \tag{4.83}$$

$$\left(1+d_S\max_{n\neq m;\{1,...,d_S\}}|\langle\psi^k_{n,t}|\psi^k_{m,t}\rangle|\right)^{d_S-1}\left(\sum_{m=1}^{s}|\langle\psi^k_{i,t}|\psi^k_{m,t}\rangle|\right) \tag{4.84}$$

Now, let us shift our attention to the terms $D^k_{s,t}$ in Theorem 4.4.1.

$$D^k_{s,t}:=\mathbf{det}\begin{pmatrix}\langle\psi^k_{1,t}|\psi^k_{1,t}\rangle & \langle\psi^k_{1,t}|\psi^k_{2,t}\rangle & \dots & \langle\psi^k_{1,t}|\psi^k_{s,t}\rangle\\ \langle\psi^k_{2,t}|\psi^k_{1,t}\rangle & \langle\psi^k_{2,t}|\psi^k_{2,t}\rangle & \dots & \langle\psi^k_{2,t}|\psi^k_{s,t}\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\psi^k_{j,t}|\psi^k_{1,t}\rangle & \langle\psi^k_{j,t}|\psi^k_{2,t}\rangle & \dots & \langle\psi^k_{s,t}|\psi^k_{s,t}\rangle\end{pmatrix} \tag{4.85}$$

Using Theorem 4.5.2 we have that

$$|D^k_{s,t}|=\left|\prod_{j=1}^{s}\left(1+\lambda_j\left(\hat{\mathbf{B}}^s_t\right)\right)\right| \tag{4.86}$$

where again

$$\hat{\mathbf{B}}^s_t:=\begin{pmatrix}0 & \langle\psi_{1,t}|\psi_{2,t}\rangle & \dots & \langle\psi_{1,t}|\psi_{s,t}\rangle\\ \langle\psi^k_{2,t}|\psi^k_{1,t}\rangle & 0 & \dots & \langle\psi^k_{2,t}|\psi^k_{s,t}\rangle\\ \vdots & \vdots & \ddots & \vdots\\ \langle\psi^k_{j,t}|\psi^k_{1,t}\rangle & \langle\psi^k_{j,t}|\psi^k_{2,t}\rangle & \dots & 0\end{pmatrix} \tag{4.87}$$

Now, using Theorem 4.5.3 we know that the eigenvalues of $\hat{B}^s_t$ lie within the Gerschgorin discs

$$\mathscr{D}^s_i:=\left\{x\in\mathbb{C}:|x|\leq\sum_{j;j\neq i}|B^s_{ij,t}|\right\}\ i\in\{1,...,k\} \tag{4.88}$$

where we have made use of the fact that $B^s_{ii,0}=0$ for all i. The superscript of $\mathscr{D}^s_i$ is used to highlight its pertinence to the determinant $D^k_{s,t}$. Now,

$$|D^k_{s,t}|=\left|\prod_{j=1}^{s}\left(1+\lambda_j\left(\hat{\mathbf{B}}^s_t\right)\right)\right|=\prod_{j=1}^{s}\left|1+\lambda_j\left(\hat{\mathbf{B}}^s_t\right)\right|\geq \tag{4.89}$$

$$\geq\prod_{j=1}^{s}\min_{x\in\mathscr{G}_s}\left|1+x\right|=\min_{x\in\mathscr{G}_s}\left|1+x\right|^s. \tag{4.90}$$

Here we remind the reader that $\mathscr{G}_s:=\bigcup_{i=1}^{s}\mathscr{D}^s_i$. Minimizing over a larger set yields a smaller minimum, hence,

$$\min_{x\in\mathscr{G}_s} \left|1+x\right|^s \geq \min_{x\in\mathscr{G}_{d_S}} \left|1+x\right|^s \geq \min_{x\in\mathscr{G}_{d_S}} \left|1-|x|\right|^k. \tag{4.91}$$

Using (4.84) and (4.91), we may now further bound the determinant-including terms in result (4.67) to obtain

$$\min_{PVM} \sum_{i=1}^{d_S} \sigma_i \left\| \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 \leq \tag{4.92}$$

$$\sum_{i=2}^{d_S} \sigma_i \sum_{s=1}^{i-1} \frac{\Big(1+d_S M_{d_s}(t)\Big)^{d_S-1} \Big(\sum_{m=1}^{s} |\langle \psi_{i,t}^k | \psi_{m,t}^k \rangle|\Big)}{\min_{x\in\mathscr{G}_{d_S}} \left|1-|x|\right|^s} = \tag{4.93}$$

$$\Big(1+d_S M_{d_S}(t)\Big)^{d_S-1} \sum_{i=2}^{d_S} \sigma_i \sum_{s=1}^{i-1} \sum_{m=1}^{s} \frac{|\langle \psi_{i,t}^k | \psi_{m,t}^k \rangle|}{\min_{x\in\mathscr{G}_{d_S}} \left|1-|x|\right|^s} \leq \tag{4.94}$$

$$\Big(1+d_S M_{d_S}(t)\Big)^{d_S-1} \sum_{i=2}^{d_S} \sigma_i \sum_{s=1}^{i-1} \sum_{m=1}^{i-1} \frac{|\langle \psi_{i,t}^k | \psi_{m,t}^k \rangle|}{\min_{x\in\mathscr{G}_{d_S}} \left|1-|x|\right|^{i-1}} \leq \tag{4.95}$$

$$\Big(1+d_S M_{d_S}(t)\Big)^{d_S-1} \sum_{i=2}^{d_S} \sigma_i (i-1) \sum_{m=1}^{i-1} \frac{|\langle \psi_{i,t}^k | \psi_{m,t}^k \rangle|}{\min_{x\in\mathscr{G}_{d_S}} \left|1-|x|\right|^{i-1}} \leq \tag{4.96}$$

$$d_S \Big(1+d_S M_{d_S}(t)\Big)^{d_S-1} \sum_{i=2}^{d_S} \sigma_i \sum_{m=1}^{i-1} \frac{|\langle \psi_{i,t}^k | \psi_{m,t}^k \rangle|}{\min_{x\in\mathscr{G}_{d_S}} \left|1-|x|\right|^{i-1}} \leq \tag{4.97}$$

$$d_S \Big(1+d_S M_{d_S}(t)\Big)^{d_S-1} \sum_{i}^{d_S} \sum_{j;j\neq i}^{d_S} \frac{\sigma_i |\langle \psi_{i,t}^k | \psi_{j,t}^k \rangle|}{\min_{x\in\mathscr{G}_{d_S}} \left|1-|x|\right|^{i-1}} \tag{4.98}$$

□

We follow up Theorem 4.5.4 with the following corollary.

Corollary 4.5.1 (Bound for small $d_S M_{d_S}(t)$)

Assume that $d_S M_{d_S}(t) < 1$, then

$$\min_{PVM} \sum_{i=1}^{d_S} \sigma_i \left\| \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1 \leq \tag{4.99}$$

$$\frac{d_S \Big(1+d_S M_{d_S}(t)\Big)^{d_S-1}}{\Big(1-d_S M_{d_S}(t)\Big)^{d_S-1}} \sum_{i=1}^{d_S} \sum_{j;j\neq i}^{d_S} \sigma_i |\langle \psi_{i,t}^k | \psi_{j,t}^k \rangle| \tag{4.100}$$

Corollary 4.5.1 and Theorem 4.5.4 give us a non-computationally heavy way of estimating the bound of Theorem 4.4.1.

4.6 Case Where $D^k_{s,t}$ Are Determinants of Circulant Matrices

Before moving on to the more general case where the density operators $\hat{\boldsymbol{\rho}}_i^{E_t^k}$ are taken to be mixtures, we will consider a special case where the bound of Theorem 4.4.1 may take a simpler form. i.e. let us return to the determinant term in (4.69), this time however we will assume that $\langle\psi^k_{i,t}|\psi^k_{j,t}\rangle$ is characterized by a function of $t|i-j|$. i.e. $f_k(t|i-j|) := \langle\psi^k_{i,t}|\psi^k_{j,t}\rangle$. Using this, we rewrite $D^k_{s,t}$ from (4.69) as follows.

$$D^k_{s,t} := \mathbf{det}\begin{pmatrix} f_k(0) & f_k(t) & \dots & f_k(t(s-1)) \\ f_k(t) & f_k(0) & \dots & f_k(t(s-2)) \\ \vdots & \vdots & \ddots & \vdots \\ f(t(j-1)) & f(t(j-2)) & \dots & f(0) \end{pmatrix} \tag{4.101}$$

(4.101) is a determinant of a circulant matrix and such determinants may be calculated exactly with ease [60]. Namely,

$$D^k_{s,t} = \prod_{m=0}^{s-1}\sum_{n=0}^{s-1} f_k(tn)e^{\frac{mn2\pi i}{s}}. \tag{4.102}$$

4.7 Mixed Environmental States

Recall that we named the sums over i in (4.65) the SQSD problem for the mixture $\sum_i \sigma_i \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}$.

$$p_E\Big\{p_i,\ \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k},\ \hat{\mathbf{P}}_i^{E_t^k}\Big\} = \sum_{i=1}^{d_S}\sigma_i Tr\Big\{\hat{\boldsymbol{\rho}}_{x_i}^{E_t^1} - \hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\Big\} \leq \sum_{i=1}^{d_S}\sigma_i\Big\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\Big\|_1 \tag{4.103}$$

where we have used the fact that $\big|Tr\{\hat{\mathbf{A}}\}\big| \leq \big\|\hat{\mathbf{A}}\big\|_1$.

The theory we have developed so far considers only the case where $\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}$ are pure states for all i and k. In this section, we will provide the analog to Theorem 4.2.2 for the case where the environmental degrees of freedom are finite mixtures of pure states. Using a simpler indexing scheme, consider a mixed state of the form $\sum_{i=1}^N p_i\hat{\boldsymbol{\rho}}_i$, where $\sum_{i=1}^N p_i = 1$ and the $\hat{\boldsymbol{\rho}}_i$ are all countably-mixed states; i.e. $\hat{\boldsymbol{\rho}}_i = \sum_{k=1}^{M_i}\eta_{ik}\hat{\boldsymbol{\rho}}_{ik}$ where all of the $\hat{\boldsymbol{\rho}}_{ik}$ are pure states and $\sum_{k=1}^{M_i}\eta_{ik} = 1$. Let us now consider the QSD problem

$$\min_{POVM}\sum_{i=1}^N p_i Tr\Big\{\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\Big\}. \tag{4.104}$$

We obtain an upper bound on (4.104) if we minimize over all PVM instead of all POVM.

$$\min_{POVM}\sum_{i=1}^N p_i Tr\Big\{\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\Big\} \leq \min_{PVM}\sum_{i=1}^N p_i Tr\Big\{\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\Big\} \leq \tag{4.105}$$

$$\min_{PVM}\sum_{i=1}^N p_i\Big\|\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{M}}_i^\dagger\Big\|_1 \tag{4.106}$$

The left-hand side of (4.105) can be bounded from below using Theorem 3.2.2 . Namely,

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j;j\neq i}^{N} p_i p_j F(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j) \leq \min_{PVM}\sum_{i=1}^{N} p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_{i=1} \hat{\mathbf{P}}_i \right\|_1 \tag{4.107}$$

Expanding the $\hat{\boldsymbol{\rho}}_i$ we see that

$$\sqrt{F(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j)} = \sqrt{F\Big(\sum_{k=1}^{M_i}\eta_{ik}\hat{\boldsymbol{\rho}}_{ik}, \sum_{k=1}^{M_j}\eta_{jk}\hat{\boldsymbol{\rho}}_{jk}\Big)} \geq \sum_{k=1}^{\min\{M_i,M_j\}} \sqrt{\eta_{ik}\eta_{jk}}\sqrt{F\Big(\hat{\boldsymbol{\rho}}_{ik}, \hat{\boldsymbol{\rho}}_{jk}\Big)} \tag{4.108}$$

where we have used Theorem 9.7 of [9] in (4.108). (4.107) now implies that

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j;j\neq i}^{N} p_i p_j \left(\sum_{k=1}^{\min\{M_i,M_j\}} \sqrt{\eta_{ik}\eta_{jk}}\sqrt{F\Big(\hat{\boldsymbol{\rho}}_{ik}, \hat{\boldsymbol{\rho}}_{jk}\Big)}\right)^2 \leq \min_{PVM}\sum_{i=1}^{N} p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{P}}_i \right\|_1 \tag{4.109}$$

This inequality shows that a necessary condition for fully solving the optimization problem (4.109) is that $F\big(\hat{\boldsymbol{\rho}}_{ik}, \hat{\boldsymbol{\rho}}_{jk}\big) = 0$ for all i, j, k where $i \neq j$. Otherwise, we run into the possibility that this minimum is bounded away from zero. For the case where the $\hat{\boldsymbol{\rho}}_i$ are not mixed states the respective relationship is $F\big(\hat{\boldsymbol{\rho}}_i, \hat{\boldsymbol{\rho}}_j\big) = 0$ for $i \neq j$, which is what we expect for the states studied in the previous section and those to be studied. For the case where the $\hat{\boldsymbol{\rho}}_i$ are finite mixtures of pure states, one will be required to analyze the fidelities between elements of any two different mixtures, say $\hat{\boldsymbol{\rho}}_i$ and $\hat{\boldsymbol{\rho}}_j$, in order to determine the discriminability of the mixture $\sum_{i=1}^{N}\hat{\boldsymbol{\rho}}_i$. We will analyze these types of mixtures in the following.

We will be estimating the right-hand side of (4.109). Our approach shall be an adaptation of the methods employed in the proof of Theorem 4.2.2 and Lemma 4.2.1. Constraining to the case where $\hat{\mathbf{P}}_i$ are projectors will yield a bound that will be useful for the cases where $F\big(\hat{\boldsymbol{\rho}}_{ik}, \hat{\boldsymbol{\rho}}_{jk}\big) = 0$ for all k when $i \neq j$ and $F\big(\hat{\boldsymbol{\rho}}_{ik}, \hat{\boldsymbol{\rho}}_{il}\big) = 0$ for all i when $l \neq k$ hold exactly and/or approximately.

We begin by noting that

$$\min_{PVM}\sum_{i=1}^{N} p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{P}}_i \right\|_1 \leq \min_{PVM}\sum_{i=1}^{N}\sum_{k=1}^{M_i} p_i\eta_{ik} \left\| \hat{\boldsymbol{\rho}}_{ik} - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_{ik} \hat{\mathbf{P}}_i \right\|_1 \tag{4.110}$$

This looks very similar to the PVM QSD problem for pure states tackled in the previous two sections (note that $p_i\eta_{ik}$ is a probability distribution, where $leqi \leq N$ and $1 \leq k \leq M_i$) except that now each element of the PVM $\big\{\hat{\mathbf{P}}_i\big\}_i$ multiplies all elements $\hat{\boldsymbol{\rho}}_{ik}$ of the ith mixture. Following the methods from the previous section, one might suggest implementing the gram-schmidt procedure once more in order to obtain an orthonormal set of vectors $\big|\phi_i\big\rangle$, one for each i. However, in this case, the operators $\hat{\boldsymbol{\rho}}_i$ are mixed and therefore do not have a representation as a vector in the corresponding Hilbert space; being able to view the mixture $\sum_i p_i\hat{\boldsymbol{\rho}}_i$ as a single index ensemble of pure states was one of the key assumptions that lead to Theorem 4.2.2.

We now assume that the $\hat{\mathbf{P}}_i$ have the following structure.

$$\hat{\mathbf{P}}_i = \sum_{k=1}^{M_i} \hat{\mathbf{P}}_{ik} \tag{4.111}$$

In order to guarantee that the $\sum_{k=1}^{M_i} \hat{\mathbf{P}}_{ik}$ is a projector, we will assume that the ranges $\hat{\mathbf{P}}_{ik}$ are orthogonal to each other, k varying from 1 to M_i.

Since all of the $\hat{\boldsymbol{\rho}}_{ik}$ are pure states, we may apply the Gram-Schmidt process in order to construct a PVM $\{\hat{\mathbf{P}}_{ik}\}_{ik}$. With the inclusion of the completion element $\mathbb{I} - \sum_i \sum_k \hat{\mathbf{P}}_{ik}$, operators $\hat{\mathbf{P}}_{ik}$ form a PVM that resolves the identity. There are $\sum_{i=1}^{N} M_i$ states $\hat{\boldsymbol{\rho}}_{ik}$, let us now arrange them in a sequence.

$$\vec{\mathscr{V}} := \Big(\hat{\boldsymbol{\rho}}_{11} \quad \dots \quad \hat{\boldsymbol{\rho}}_{1M_1} \quad \hat{\boldsymbol{\rho}}_{21} \quad \dots \quad \hat{\boldsymbol{\rho}}_{2M_2} \quad \dots \quad \hat{\boldsymbol{\rho}}_{N1} \quad \dots\dots \quad \hat{\boldsymbol{\rho}}_{NM_N}\Big). \tag{4.112}$$

Let us name the sth component of this sequence $\mathscr{V}_s := |\xi_s\rangle\langle\xi_s|$. Since the cardinality of the sequence (4.112) is the same as the cardinality of the set $\{\hat{\boldsymbol{\rho}}_{ik}\}_{ik}$, $(1 \leq i \leq N, 1 \leq k \leq M_i)$, there exists a bijection $\mathscr{M}$ mapping every vector (i,k) (which we write in shorthand as ik) to a unique s and viceversa.

Assuming that the $|\xi_s\rangle\langle\xi_s|$ form a linearly independent set we now apply the Gram-Schmidt process to obtain the family of orthonormal states

$$|\phi_1\rangle := |\xi_1\rangle \tag{4.113}$$

$$|\phi_s\rangle := \frac{1}{\alpha_s}\Big\{|\xi_s\rangle - \sum_{k=1}^{s-1} \langle\phi_k|\xi_s\rangle|\phi_k\rangle\Big\},\ \ s \in \{1,2,...,\sum_{i=1}^{N} M_i\} \tag{4.114}$$

and as before $\alpha_i := \big\||\xi_i\rangle - \sum_{k=1}^{i-1} \langle\phi_k|\xi_i\rangle|\phi_k\rangle\big\| = \sqrt{1 - \sum_{k=1}^{i-1} |\langle\phi_k|\xi_i\rangle|^2}$ for $i > 1$ and $\alpha_1 = 1$ are the respective normalization constants. An identity resolving PVM $\Big\{|\phi_s\rangle\langle\phi_s|\Big\}_s \bigcup \Big\{\mathbb{I} - \sum_s |\phi_s\rangle\langle\phi_s|\Big\}$ has thus been constructed, defining $\omega_s := p_i\eta_{ik}$, where $s = \mathscr{M}\big((i,k)\big)$, we may now rewrite and bound

$$\sum_{i=1}^{N}\sum_{k=1}^{M_i} p_i\eta_{ik}\Big\|\hat{\boldsymbol{\rho}}_{ik} - \mathbf{P}_i\hat{\boldsymbol{\rho}}_{ik}\mathbf{P}_i\Big\|_1 \tag{4.115}$$

as follows.

$$\sum_s \omega_s \Big\||\xi_s\rangle\langle\xi_s| - \Big(\sum_{\substack{l \ni \\ proj_{e_1}(\mathscr{M}^{-1}(l)) = \\ proj_{e_1}(\mathscr{M}^{-1}(s))}} |\phi_l\rangle\langle\phi_l|\Big)|\xi_s\rangle\langle\xi_s|\Big(\sum_{\substack{l \ni \\ proj_{e_1}(\mathscr{M}^{-1}(l)) = \\ proj_{e_1}(\mathscr{M}^{-1}(s))}} |\phi_l\rangle\langle\phi_l|\Big)\Big\|_1 = \tag{4.116}$$

$$\sum_s \omega_s \Big\||\xi_s\rangle\langle\xi_s| - |\phi_s\rangle\langle\phi_s|\xi_s\rangle\langle\xi_s|\phi_s\rangle\langle\phi_s| - \Big(\sum_{\substack{l \neq s \ni \\ proj_{e_1}(\mathscr{M}^{-1}(l)) = \\ proj_{e_1}(\mathscr{M}^{-1}(s))}} |\phi_l\rangle\langle\phi_l|\Big)|\xi_s\rangle\langle\xi_s|\Big(\sum_{\substack{l \neq s \ni \\ proj_{e_1}(\mathscr{M}^{-1}(l)) = \\ proj_{e_1}(\mathscr{M}^{-1}(s))}} |\phi_l\rangle\langle\phi_l|\Big)\Big\|_1 \leq \tag{4.117}$$

$$\sum_s \omega_s \left\| \left|\xi_s\right\rangle\left\langle\xi_s\right| - \left|\phi_s\right\rangle\left\langle\phi_s\middle|\xi_s\right\rangle\left\langle\xi_s\middle|\phi_s\right\rangle\left\langle\phi_s\right| \right\|_1 + \sum_s \omega_s \left\| \left(\sum_{\substack{l \neq s \ni \\ proj_{e_1}\left(\mathscr{M}^{-1}(l)\right) = \\ proj_{e_1}\left(\mathscr{M}^{-1}(s)\right)}} \left|\phi_l\right\rangle\left\langle\phi_l\right| \right) \left|\xi_s\right\rangle\left\langle\xi_s\right| \left(\sum_{\substack{l \neq s \ni \\ proj_{e_1}\left(\mathscr{M}^{-1}(l)\right) = \\ proj_{e_1}\left(\mathscr{M}^{-1}(s)\right)}} \left|\phi_l\right\rangle\left\langle\phi_l\right| \right) \right\|_1 \leq \tag{4.118}$$

$$2\sum_s \omega_s \sum_{k=1}^{s-1} \left|\left\langle\phi_k\middle|\xi_s\right\rangle\right| + \sum_s \omega_s \sum_{\substack{l \neq s \ni \\ proj_{e_1}\left(\mathscr{M}^{-1}(l)\right) = \\ proj_{e_1}\left(\mathscr{M}^{-1}(s)\right)}} \left(\sum_{\substack{k \neq s \ni \\ proj_{e_1}\left(\mathscr{M}^{-1}(l)\right) = \\ proj_{e_1}\left(\mathscr{M}^{-1}(s)\right)}} \left\| \left|\phi_l\right\rangle\left\langle\phi_l\middle|\xi_s\right\rangle\left\langle\xi_s\middle|\phi_k\right\rangle\left\langle\phi_k\right| \right\|_1 \right) = \tag{4.119}$$

$$2\sum_s \omega_s \sum_{k=1}^{s-1} \left|\left\langle\phi_k\middle|\xi_s\right\rangle\right| + \sum_s \omega_s \sum_{\substack{l \neq s \ni \\ proj_{e_1}\left(\mathscr{M}^{-1}(l)\right) = \\ proj_{e_1}\left(\mathscr{M}^{-1}(s)\right)}} \left(\sum_{\substack{k \neq s \ni \\ proj_{e_1}\left(\mathscr{M}^{-1}(l)\right) = \\ proj_{e_1}\left(\mathscr{M}^{-1}(s)\right)}} \left|\left\langle\phi_l\middle|\xi_s\right\rangle\left\langle\xi_s\middle|\phi_k\right\rangle\right| \right) \tag{4.120}$$

where we have used Lemma 4.2.1 in going from (4.118) to (4.119). Using Lemma 4.2.2 we may explicitly write the terms $|\langle\phi_l|\xi_s\rangle|$ as Gram-Schmidt determinants and use these to estimate the efficacy of the PVM built from (4.113) and (4.114). (4.120) may be further bounded by the following term.

$$(4.120) \leq 3\sum_s \omega_s \sum_{l;l\neq s} \left|\left\langle\phi_l\middle|\xi_s\right\rangle\right| \tag{4.121}$$

where the only restriction on the sums is that $l \neq s$. As already mentioned, this may be better estimated using Lemma 4.2.2. We state the result (4.120) as a theorem.

Theorem 4.7.1

Consider a mixed state of the form $\sum_{i=1}^{N} p_i \hat{\boldsymbol{\rho}}_i$, $\sum_{i=1}^{N} p_i = 1$, where $\hat{\boldsymbol{\rho}}_i$ are all countable mixtures of pure states,i.e. $\hat{\boldsymbol{\rho}}_i = \sum_{k=1}^{M_i} \eta_{ik} |\psi_{ik}\rangle\langle\psi_{ik}|$ where $\sum_{k=1}^{M_i} \eta_{ik} = 1$. Furthermore, assume that the set $\{|\psi_{ik}\rangle\}_{ik}$ is linearly independent. Then

$$\min_{PVM} \sum_{i=1}^{N} p_i \left\| \hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i \hat{\boldsymbol{\rho}}_i \hat{\mathbf{P}}_i \right\|_1 \leq \tag{4.122}$$

$$2 \sum_s \omega_s \sum_{k=1}^{s-1} |\langle\phi_k|\xi_s\rangle| + \sum_s \omega_s \sum_{\substack{l \neq s \ni \\ proj_{e_1}\left(\mathscr{M}^{-1}(l)\right) = \\ proj_{e_1}\left(\mathscr{M}^{-1}(s)\right)}} \left(\sum_{\substack{k \neq s \ni \\ proj_{e_1}\left(\mathscr{M}^{-1}(l)\right) = \\ proj_{e_1}\left(\mathscr{M}^{-1}(s)\right)}} |\langle\phi_l|\xi_s\rangle\langle\xi_s|\phi_k\rangle| \right) \tag{4.123}$$

where

$$|\xi_s\rangle := |\psi_{\mathscr{M}((i,k))}\rangle \tag{4.124}$$

$$\mathscr{M} \quad (defined\ in\ the\ discussion\ following\ (4.112)) \tag{4.125}$$

$$|\phi_1\rangle := |\xi_1\rangle \tag{4.126}$$

$$|\phi_s\rangle := \frac{1}{\alpha_s} \left\{ |\xi_s\rangle - \sum_{k=1}^{s-1} \langle\phi_k|\xi_s\rangle |\phi_k\rangle \right\}, \ s \in \{1, 2, ..., S := \sum_{i=1}^{N} M_i\} \tag{4.127}$$

and

$$\alpha_i := \left\| |\xi_i\rangle - \sum_{k=1}^{i-1} \langle\phi_k|\xi_i\rangle |\phi_k\rangle \right\| = \sqrt{1 - \sum_{k=1}^{i-1} |\langle\phi_k|\xi_i\rangle|^2} \tag{4.128}$$

for $i > 1$ and $\alpha_1 = 1$ are the respective normalization constants.

Proof. The proof may be found in the preceding discussion. □

As a final remark, we point out that a more general mixture of non-pure states $\sum_i p_i \hat{\boldsymbol{\rho}}_i$ may be obtained by considering the case where $\boldsymbol{\rho}_i := \mathscr{E}_i(\hat{\boldsymbol{\rho}}_0)$ ($\hat{\boldsymbol{\rho}}_0$ is a pure state); $\mathscr{E}_i$ being arbitrary quantum maps for all i. In general the $\boldsymbol{\rho}_i := \mathscr{E}_i(\hat{\boldsymbol{\rho}}_\mathbf{0})$ will not be expressable as finite mixtures. If such a case is encountered we may use (4.120) only if the $\boldsymbol{\rho}_i := \mathscr{E}_i(\hat{\boldsymbol{\rho}}_\mathbf{0})$ may be approximated by countable mixtures. In Chapter 5 we will study a case where a mixture of the type $\sum_i p_i \mathscr{E}_i(\hat{\boldsymbol{\rho}}_0)$ is encountered. However, for some of the cases to be studied in Chapter 5, the $\mathscr{E}_i$ will be approximately unitary maps and so we use this to approximate $\sum_i p_i \mathscr{E}_i(\hat{\boldsymbol{\rho}}_0)$ with a countable mixture of pure states. More general cases are still open to further investigation.

4.8 How General May the $\hat{\mathbf{B}}_k$ Be?

We conclude this subsection with the following Theorem. The question it sheds light on is the following: "How general may $\hat{\mathbf{B}}$ whilst still inducing dynamics (4.46) which are convergent to and SBS state?"

Theorem 4.8.1 (Sufficient conditions for the convergence to SBS for a broad family of multipartite states)

Consider the setup spanning equations (4.38) through (4.46). If for all k, $\hat{\mathbf{B}}_k$ has a non-empty Rajchman subspace $\mathscr{H}_{E^k,rc}$ ([58]), and $\hat{\boldsymbol{\rho}}_0^{E^k}$ is a finite mixture of pure states in $\mathcal{S}(\mathscr{H}_{E^k,rc})$, then $\hat{\boldsymbol{\rho}}$ converges asymptotically in $t>0$ to an SBS state with respect to the trace norm topology.

Proof. Using Theorems 4.4.1 and 4.7.1, $\sum_{k=1}^{N_E}\sum_{i=1}^{d_S}\sigma_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}-\hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right\|_1$ may be estimated using inner products of distinct $\left|\psi_{ik}\right\rangle$ (see (4.34) and (4.33)). Furthermore, for all $i\neq j$, $\Gamma(i,j,t)$ (4.45) is a product of inner products. The inner products featured in both diagonal terms

$$\sum_{k=1}^{N_E}\sum_{i=1}^{d_S}\sigma_i\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}-\hat{\mathbf{P}}_i^{E_t^k}\hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\hat{\mathbf{P}}_i^{E_t^k}\right\|_1$$

and the off-diagonal terms $\sum_i\sum_{j;j\neq i}^{d_S}\left|\sigma_{i,j}\Gamma(i,j,t)\right|$ have the structure $\left\langle\psi\middle|e^{-it\alpha\hat{\mathbf{B}}_k}\middle|\phi\right\rangle$ with $\left|\psi\right\rangle\in\mathscr{H}_{E^k,rc}$ which implies the claim (using the fact that the Rajchman subspace is a reducing subsapce). □

Chapter 5

SBS for Continuous Variables

In this Chapter, we will generalize the concept of SBS to the case where the operator $\hat{\mathbf{X}}$ is a position operator acting in $\mathscr{H}_S = L^2(\mathbb{R})$. Whilst studying SBS theory in such a case, we shall be interested in the decoherence of superpositions of the general eigenstates of $\hat{\mathbf{X}}$. In particular, let $\hat{\boldsymbol{\rho}}_{S_0}$ be state in $\mathcal{S}(L^2(\mathbb{R}))$, we will be keen on studying the behaviour of

$$\langle x' | \mathscr{E}_t(\hat{\boldsymbol{\rho}}_S) | x \rangle \tag{5.1}$$

as t becomes large within a specified time frame of interest, where $\mathscr{E}_t$ is a quantum map that produces decoherence effects [12]. Notice the continuous nature of the coherence terms (5.1), i.e. for any $x, x' \in \mathbb{R}$ (5.1) is a coherence term vs the analogous case studied in the previous chapter, i.e. (4.49), where there was a countable amount of coherence terms. We henceforth will be referring to the case where $\hat{\mathbf{X}}$ has purely continuous spectrum as the SBS theory for continuous variables (SBSCV).

Let us assume the *quantum-measurement limit*, hypothesis discussed in Subsection 1.6.3 with $\mathbf{dim}(\mathscr{H}_S) = \infty$ and $\mathbf{dim}(\mathscr{H}_{E^k}) = \infty$ for all k. i.e.

$$\hat{\mathbf{H}}_{tot} \approx \hat{\mathbf{H}}_{int} \tag{5.2}$$

We will also assume an interaction Hamiltonian of the von Neumann type. Hence,

$$\hat{\mathbf{H}}_{int} = \hat{\mathbf{X}} \otimes \sum_{k=1}^{N} g_k \hat{\mathbf{B}}_k \tag{5.3}$$

We already mentioned that $\hat{\mathbf{X}}$ will be taken to be the position operator for now. The $\hat{\mathbf{B}}_k$ will, in general, be taken to be self-adjoint operator with a non-empty Rajchman Subspace (Theorem 3.5.1); each acting in its respective Hilbert space, i.e. all of the $\hat{\mathbf{B}}_k$ act on different Hilbert spaces. An example of the latter, and one which we will explore in-depth, is the case where all of the $\hat{\mathbf{B}}_k$ are position or momentum operators. The time evolution operator corresponding to (5.3) is the following.

$$\hat{\mathbf{U}}_t = e^{-it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N} g_k\hat{\mathbf{B}}_k}. \tag{5.4}$$

Considering a product state as our initial state, as we did in (4.38), acting on the appropriate product Hilbert space, we apply the time evolution operator (5.4).

$$\hat{\boldsymbol{\rho}}_t = \Big(e^{-it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N} g_k\hat{\mathbf{B}}_k}\Big)\Big(\hat{\boldsymbol{\rho}}_{S_0}\otimes\bigotimes_{k=1}^{N}\hat{\boldsymbol{\rho}}^{E_0^k}\Big)\Big(e^{it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N} g_k\hat{\mathbf{B}}_k}\Big). \tag{5.5}$$

In order to study the state of the subsystem formed by the system S and the first N_E environments, we take the partial trace of the time-evolved density operator over the remaining $M_E := N - N_E$ environments. Using Lemma 1.6.1 and (5.5), the following state is the result of partially tracing M_E environments from (5.5).

$$\mathscr{U}_{N_E,t}\Big(\mathscr{E}_t^{M_E}(\hat{\boldsymbol{\rho}}_{S_0})\otimes\bigotimes_{k=1}^{N_E}\hat{\boldsymbol{\rho}}^{E_0^k}\Big) \tag{5.6}$$

Where

$$\mathscr{U}_{n,t}(\hat{\mathbf{A}}) := e^{-it\hat{\mathbf{X}}\otimes\hat{\mathbf{S}}_n}(\hat{\mathbf{A}})e^{it\hat{\mathbf{X}}\otimes\hat{\mathbf{S}}_n} \tag{5.7}$$

$$\hat{\mathbf{S}}_n := \sum_{k=1}^{n} g_k\hat{\mathbf{B}}_k \tag{5.8}$$

and

$$\mathscr{E}_t^{M_E}\{\hat{\boldsymbol{\sigma}}\} := \int\int\langle x|\hat{\boldsymbol{\sigma}}|y\rangle\Gamma_{M_E}(t,x,y)|x\rangle\langle y|dxdy. \tag{5.9}$$

where

$$\Gamma_{M_E}(t,x,y) := \prod_{k=N_E+1}^{N} Tr_k\Big\{\big(e^{-itxg_k\hat{\mathbf{B}}_k}\big)\hat{\boldsymbol{\rho}}^{E_0^k}\big(e^{ityg_k\hat{\mathbf{B}}_k}\big)\Big\} \tag{5.10}$$

$M_E = N - N_E$, the number of traces being taken in equation (5.10). To simplify the notation, we shall forgo all but two environments, i.e. $N = 2$, $N_E = M_E = 1$. Generalizing everything to a general M_E, N_E and N is trivial. In such a case, after partial tracing over one of the environments we obtain the following density operator.

$$\hat{\boldsymbol{\rho}}_t := \mathscr{U}_{1,t}\big(\mathscr{E}_t^1\{\hat{\boldsymbol{\rho}}_{S_0}\}\otimes\hat{\boldsymbol{\rho}}^{E_0^1}\big). \tag{5.11}$$

The map $\mathscr{E}_t^1$ is a decoherence quantum map and $\mathscr{U}_{1,t}$ is a unitary map obtained from the Hamiltonian (5.3) for the case $N = 2$. Again, all of the ensuing results may be easily generalized to a general N_E.

The primary divergence from the techniques presented in the previous section will be the necessity to partition the operator $\mathscr{E}_t^1\{\hat{\boldsymbol{\rho}}_{S_0}\}$. For the case where $\hat{\mathbf{X}}$ is a position operator, the partitions of interest are those of the following.

$$\mathscr{E}_t^1\{\hat{\boldsymbol{\rho}}_{S_0}\} = \sum_i\sum_j\hat{\mathbf{P}}_{\Delta_{i,t}}\mathscr{E}_t^1\{\hat{\boldsymbol{\rho}}_{S_0}\}\hat{\mathbf{P}}_{\Delta_{j,t}} \tag{5.12}$$

where the operators $\hat{\mathbf{P}}_{\Delta_{i,t}}$ are projector operators defined as follows. $\hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\mathbf{X}} = \chi_{\Delta_{i,t}}(\hat{\mathbf{X}})$ ($\chi_{\Delta_{i,t}}(x)$ are indicator functions and the $\Delta_{i,t}$ are subsets of the real line). This is akin to what was done in the previous section to obtain (4.40), where in lieu of the projectors $\hat{\mathbf{P}}_{\Delta_{i,t}}$, projectors onto the eigen-subspspaces corresponding to the eigenvectors $\{|i\rangle\}_{i=1}^{d_S}$ are used (see discussion following (4.40)). Although using projectors onto the generalized eigensubspaces of $\hat{\mathbf{X}}$ (now a position operator) is the most natural means of generalizing (4.40), there are limitations. The projectors $\hat{\mathbf{P}}_{\Delta_{i,t}}$ will be in general time-dependent, and the size of the $\Delta_{i,t}$ will be restricted by quantum-metrological and other physical limitations [15]. We will not explore the quantum-metrological aspects of SBSCV in this work; we simply highlight the fact that the size of the magnitudes of the $\Delta_{i,t}$ may be bounded from below and from above by parameters depending on quantum-metrological physical limitations. A physical interpretation of the limiting smallness of the subspaces $\Delta_{i,t,}$ may be deduced within the context of von Neumann's theory of quantum measurement (2.2.3). The set of $\hat{\mathbf{P}}_{\Delta_{i,t}}$ is a PVM and therefore characterizes a von Neumann measurement on the system S. With the latter in mind, the sizes of the $\Delta_{i,t}$ may be interpreted as resolution limits. Indeed, resolving the position of an arbitrarily small particle would require arbitrarily larger amounts of energy as the size of the particle becomes smaller. Due to the technological limitations of monitoring apparatuses, there will always be a limit to the smallness of the resolution $\Delta_{i,t}$. When introducing the approximate SBS state for continuous variables, a specific PVM acting on the system S will be assumed for every t prior to estimating the respective optimization problem that ensues (see 5.19). It is there where the partition (5.12) will play a key role.

Assuming that we have an appropriate partition (5.12), we may now mirror our work from the previous Chapter in order to define an appropriate SBS for the CV case. We then develop tools to study the convergence of (5.12) to such an SBS state in t. First, we present a definition that generalizes Definition 4.1.2 to a definition that supports the CV setting. Namely,

Definition 5.0.1 (SBS, a more general definition)

Let $\mathscr{H}_S \otimes \bigotimes_{k=1}^{N_E} \mathscr{H}_{E^k}$ be some tensor product Hilbert space. $\mathscr{H}_S$ will correspond to the system while $\mathscr{H}_{E^1}, \mathscr{H}_{E^2}, ..., \mathscr{H}_{E^{N_E}}$ will all correspond to environmental degrees of freedom. A SBS state acting in $\mathcal{S}\big(\mathscr{H}_S \otimes \bigotimes_{k=1}^{N_E} \mathscr{H}_{E^k}\big)$ is any density operator of the form

$$\hat{\boldsymbol{\rho}}_{SBS} := \sum_i \left(\hat{\mathbf{P}}_{S_i} \otimes \bigotimes_{k=1}^{N_E} \mathbb{I}_{E^k}\right) \hat{\boldsymbol{\rho}} \left(\hat{\mathbf{P}}_{S_i} \otimes \bigotimes_{k=1}^{N_E} \mathbb{I}_{E^k}\right) \tag{5.13}$$

satisfying the following properties.

Property 1)

$$F\big(\hat{\boldsymbol{\rho}}_i^{E^k}, \hat{\boldsymbol{\rho}}_j^{E^k}\big) = 0 \;\; \forall \;\; i \neq j \tag{5.14}$$

where $\big\{\hat{\mathbf{P}}_{S_i}\big\}_i$ is a PVM acting in $\mathscr{H}_S$ and

$$\hat{\boldsymbol{\rho}}_i^{E^k} := T_S\left\{T_{E_{k'\neq k}}\left\{\left(\hat{\mathbf{P}}_{S_i} \otimes \bigotimes_{k=1}^{N_E} \mathbb{I}_{E^k}\right) \hat{\boldsymbol{\rho}} \left(\hat{\mathbf{P}}_{S_i} \otimes \bigotimes_{k=1}^{N_E} \mathbb{I}_{E^k}\right)\right\}\right\} \tag{5.15}$$

($Tr_{E^{k'\neq k}}$ means that we trace out over all environments with the exception of the kth environment).

Property 2) $Tr_S\{\hat{\boldsymbol{\rho}}\}$ is a separable state. i.e. it is of the form

$$Tr_S\{\hat{\boldsymbol{\rho}}\} = \sum_i p_i \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_i^{E^k} \;\; \Big(\sum_i p_i = 1\Big) \tag{5.16}$$

or

$$Tr_S\{\hat{\boldsymbol{\rho}}\} = \int p(x) \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_x^{E^k} dx \;\; \Big(\int p(x) dx = 1\Big) \tag{5.17}$$

Indeed, Definition 4.1.2 satisfies the properties of Definition 5.0.1. Furthermore, for every $t > 0$ one may deduce an approximate SBS state, in the sense of Definition 5.0.1 from $\hat{\boldsymbol{\rho}}_t$ (from 5.11) as follows. Let the PVM $\big\{\hat{\mathbf{P}}_{\Delta_{i,t}}\big\}_i$ and $\big\{\hat{\mathbf{P}}_i^{E_t^1}\big\}_i$ be PVM characterizing von Neumann measurements for the system S and the environment E^1 respectively. We may use the PVM in (5.12) acting on S for generating a partition (5.12) as the PVM associated with the von Neumann measurement being made on S at time t. The post-measurement joint state of the system and environment are expected to be in agreement with respect to the value of i, we force this as follows;

$$\frac{1}{\mathcal{N}(t)} \sum_i \big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1}\big) \hat{\boldsymbol{\rho}}_t \big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1}\big) \tag{5.18}$$

where $\mathscr{N}(t)$ is a normalization constant. The latter is an SBS state which approximates $\hat{\boldsymbol{\rho}}_t$ at time t.

To get the SBS state, constructed via the algorithm described in the previous paragraph, closest (in the trace distance sense) to $\hat{\boldsymbol{\rho}}_t$ for a fixed $t > 0$ we must solve the optimization problem

$$\min_{PVM} \frac{1}{2}\left\|\hat{\boldsymbol{\rho}}_t - \frac{1}{\mathscr{N}(t)}\sum_i (\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\hat{\boldsymbol{\rho}}_t(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\right\|_1 \tag{5.19}$$

where the minimum is taken over all PVM acting on the environmental degree of freedom. In general, it will not be possible to solve these sorts of optimization problems (5.19). We will only be interested in the asymptotic behavior of (5.19) with respect to t, hence we seek only to bound (5.19) by something resembling (4.65) in the previous chapter for the analogous problem in the discrete variables case.

Making strides towards an analog of (4.65) for the CV case we bound (5.19) as follows.

$$\min_{PVM} \frac{1}{2}\left\|\hat{\boldsymbol{\rho}}_t - \frac{1}{\mathscr{N}(t)}\sum_i (\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\hat{\boldsymbol{\rho}}_t(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\right\|_1 = \tag{5.20}$$

$$\min_{PVM} \frac{1}{2}\left\|\sum_i\sum_j \hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_{j,t}} - \frac{1}{\mathscr{N}(t)}\sum_i (\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\hat{\boldsymbol{\rho}}_t(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\right\|_1 \tag{5.21}$$

which in turn may be bounded by two terms as follows.

$$\min_{PVM} \frac{1}{2}\left\|\sum_i\sum_j \hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_{j,t}} - \frac{1}{\mathscr{N}(t)}\sum_i (\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\hat{\boldsymbol{\rho}}_t(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\right\|_1 \leq \tag{5.22}$$

$$\min_{PVM} \left(\frac{1}{2}\left\|\sum_i \hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_{i,t}} - \frac{1}{\mathscr{N}(t)}\sum_i (\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\hat{\boldsymbol{\rho}}_t(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\right\|_1 + \right. \tag{5.23}$$

$$\left. \frac{1}{2}\left\|\sum_i\sum_{j;j\neq i} \hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_{j,t}}\right\|_1\right) = \tag{5.24}$$

$$\min_{PVM} \left(\frac{1}{2}\left\|\sum_i \hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_{i,t}} - \frac{1}{\mathscr{N}(t)}\sum_i (\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\hat{\boldsymbol{\rho}}_t(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t^1})\right\|_1\right) + \tag{5.25}$$

$$\frac{1}{2}\left\|\sum_i\sum_{j;j\neq i} \hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_{j,t}}\right\|_1 \tag{5.26}$$

The approach we present here for tackling the SBSCV problem is that of studying the bound above consisting of the terms (5.25) and (5.26). In what follows we will dedicate a separate section to each of the terms (5.25) and (5.26) respectively. We will refer to the term (5.25) as the *diagonal term* and to the term (5.26) as the *off-diagonal term* (aka the coherence term). Before getting into the fact of the matter, we will briefly comment on the main mathematical difficulty arising in SBSCV theory; motivating the partition (5.12). We will then present useful bounds that will aid in studying the diagonal and off-diagonal terms (5.25) and (5.26) respectively.

5.1 Problem With Definition 4.1.2 When Introducing Continuous Variables

With this section, we hope to shed light on our reasoning behind the new definition for SBS presented in Definition 5.0.1.

There are dire hurdles that arise when attempting to define an SBS state for the case where continuous variables are involved. To appreciate them, let us examine the state (5.11) in such a case. The system's state is now a density operator $\hat{\boldsymbol{\rho}}_{S_0}$ in an infinite-dimensional Hilbert space; for our purposes, it will be convenient to take this space to be $L^2(\mathbb{R})$. Analogously to (4.36), we define the interaction of the system with the environment as

$$H_{int} = \gamma \hat{\mathbf{X}} \otimes \hat{\mathbf{B}} \tag{5.27}$$

for simplicity; where $\hat{\mathbf{X}}$ is the position operator. Being a trace-class operator, $\hat{\boldsymbol{\rho}}_{S_0}$ can be represented as an integral operator, whose kernel we denote by $K(x, y)$. The expansion analogous to (4.40) is the following:

$$\hat{\boldsymbol{\rho}}_t = \int \int K(x, y) \gamma_{x,y}^2(t) |x\rangle\langle y| \otimes \hat{\boldsymbol{\rho}}_{x,y}^{E_t^1} dx dy \tag{5.28}$$

where as expected $\hat{\boldsymbol{\rho}}_{x,y}^{E_t^1} := e^{-ix\gamma\hat{\mathbf{B}}t} \hat{\boldsymbol{\rho}}^{E_0^1} e^{iy\gamma\hat{\mathbf{B}}t}$ and $\gamma_{x,y}^2(t) := Tr\{\hat{\boldsymbol{\rho}}_{x,y}^{E_t^2}\}$. Unlike the state (4.40), the state (5.28) does not have a clear decomposition into diagonal and off-diagonal terms using the spectral decomposition of the operator $\hat{\boldsymbol{X}}$ in terms of generalized eigenvectors $|x\rangle$, which we have employed to expand $\mathscr{U}_t\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\big) \otimes \hat{\boldsymbol{\rho}}^{E_0}\big) = \big(e^{-it\gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}}\big)\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\big) \otimes \hat{\boldsymbol{\rho}}^{E_0}\big)\big(e^{-it\gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}}\big)$; herein we have dropped the subscripts and superscripts indicating that we have traced over one environmental degree of freedom per the notional conventions prescribed in equations (5.6) through (5.10). We shall forgo usage of such superscripts and subscripts from now on unless the values of M_E and N_E are relevant.

In the finite-dimensional case, we could clearly distinguish between diagonal and off-diagonal entries in order to deduce an SBS structure approximating the state in question (see the work leading to 4.65). In the continuous variable case, this approach breaks down since the diagonal term is now

$$\hat{\boldsymbol{\rho}}_t = \int K(x, x) |x\rangle\langle x| \otimes \hat{\boldsymbol{\rho}}_x^{E_t} dx \tag{5.29}$$

which is not a trace class operator, since it is unitarily equivalent to a tensor product of a multiplication operator and a trace class operator—thus it cannot represent a quantum state. Being able to separate between diagonal and off-diagonal terms in Chapter 4 was a key step in our estimation process (what led to 4.65), to proceed similarly for the CV case we must partition the state (5.28)by applying the partition (5.12) to the system's degree of freedom.

Another difficulty in moving into the continuous variable case is an increase in complexity when dealing with trace norms; starting from the fact that $\big\| |x\rangle\langle y| \big\|_1$ is undefined for generalized states $|x\rangle$ and $|y\rangle$.

5.2 Partitioning (5.11)

To formally introduce our approach for the study of SBS in the CV case we will first discuss the phenomenon of decoherence and its ramifications to our model (5.11). In this case, decoherence arises from the quantum map $\mathscr{E}_t$ in:

$$\hat{\boldsymbol{\rho}}_t = \big(e^{-it\gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}}\big)\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\big)\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\big(e^{it\gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}}\big) \tag{5.30}$$

For the remainder of this chapter we will be assuming that the states $\hat{\boldsymbol{\rho}}_{S_0}$ and $\hat{\boldsymbol{\rho}}^{E_0}$ are pure. As we have done in (5.28), we use the representation

$$\hat{\boldsymbol{\rho}}_{S_0} = \int\int K(x,y)|x\rangle\langle y|dxdy \tag{5.31}$$

using the generalized eigenvectors of the position operator $\hat{\mathbf{X}}$. Using representation (5.31), and refering back to (5.9), the effect of $\mathscr{E}_t$ on $\hat{\boldsymbol{\rho}}_{S_0}$ is hence

$$\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\big) = \int\int K(x,y)\Gamma(t,x,y)\big|x\big\rangle\big\langle y\big|dxdy \tag{5.32}$$

where $\Gamma(t,x,y)$ is a kernel yielding non-unitary dynamics obtained via partial tracing as seen in (5.10). Substituting this into (5.30) we obtain

$$\hat{\boldsymbol{\rho}}_t = \int\int K(x,y)\Gamma(t,x,y)\big|x\big\rangle\big\langle y\big|\otimes\hat{\boldsymbol{\rho}}^{E_t}_{x,y}dxdy \tag{5.33}$$

where we remind the reader that $\hat{\boldsymbol{\rho}}^{E_t}_{x,y} := e^{-it\gamma x\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}^{E_0}e^{it\gamma y\hat{\mathbf{B}}}$.

For fixed $t > 0$ we adopt a partition characterized by a PVM $\hat{\mathbf{P}}_{\Delta_{i,t}} := \chi_{\Delta_{i,t}}\big(\hat{\mathbf{X}}\big)$ acting on the degree of freedom pertaining to the system, as was done in (5.12), in order to express (5.33) as follows.

$$\hat{\boldsymbol{\rho}}_t = \sum_i\sum_j \hat{\mathbf{P}}_{\Delta_{i,t}}\Big(\int\int K(x,y)\Gamma(t,x,y)\big|x\big\rangle\big\langle y\big|\otimes\hat{\boldsymbol{\rho}}^{E_t}_{x,y}dxdy\Big)\hat{\mathbf{P}}_{\Delta_{j,t}} = \tag{5.34}$$

$$\sum_i\sum_j\Big(\int\int K(x,y)\Gamma(t,x,y)\hat{\mathbf{P}}_{\Delta_{i,t}}\big|x\big\rangle\big\langle y\big|\hat{\mathbf{P}}_{\Delta_{j,t}}\otimes\hat{\boldsymbol{\rho}}^{E_t}_{x,y}dxdy\Big) = \tag{5.35}$$

$$\sum_i\sum_j\int_{\Delta_{i,t}}\int_{\Delta_{j,t}} K(x,y)\Gamma(t,x,y)|x\rangle\langle y|\otimes\hat{\boldsymbol{\rho}}^{E_t}_{x,y}dxdy \tag{5.36}$$

Once again, we call the elements of the sum (5.36) for which $i \neq j$ the "off-diagonal terms" and the terms for which $i = j$ "the diagonal terms".

5.3 Estimating the "Off-diagonal Terms" (5.26)

Given a multipartite of the form prescribed in (5.36), the off-diagonal terms, i.e. $i \neq j$, may be estimated as follows. Focusing on (5.36) with $i \neq j$, some elementary work leads to

$$\left\| \sum_i \sum_{j;j\neq i} \int_{\Delta_{i,t}} \int_{\Delta_{j,t}} K(x,y)\Gamma(t,x,y)|x\rangle\langle y| \otimes \hat{\boldsymbol{\rho}}^{E_t}_{x,y} dxdy \right\|_1 = \tag{5.37}$$

$$\left\| e^{-it\gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}} \left(\left(\sum_i \sum_{j;j\neq i} \int_{\Delta_{i,t}} \int_{\Delta_{j,t}} K(x,y)\Gamma(t,x,y)|x\rangle\langle y| dxdy \right) \otimes \hat{\boldsymbol{\rho}}^{E_0} \right) e^{it\gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}} \right\|_1 = \tag{5.38}$$

$$\left\| \left(\sum_i \sum_{j;j\neq i} \int_{\Delta_{i,t}} \int_{\Delta_{j,t}} K(x,y)\Gamma(t,x,y)|x\rangle\langle y| dxdy \right) \otimes \hat{\boldsymbol{\rho}}^{E_0} \right\|_1 = \tag{5.39}$$

$$\left\| \sum_i \sum_{j;j\neq i} \int_{\Delta_{i,t}} \int_{\Delta_{j,t}} K(x,y)\Gamma(t,x,y)|x\rangle\langle y| dxdy \right\|_1 \leq \tag{5.40}$$

$$\sum_i \sum_{j;j\neq i} \left\| \int_{\Delta_{i,t}} \int_{\Delta_{j,t}} K(x,y)\Gamma(t,x,y)|x\rangle\langle y| dxdy \right\|_1 = \tag{5.41}$$

$$\sum_i \sum_{j;j\neq i} \left\| \hat{\mathbf{P}}_{\Delta_{i,t}} \mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0}) \hat{\mathbf{P}}_{\Delta_{j,t}} \right\|_1 \tag{5.42}$$

where again $\hat{\mathbf{P}}_{\Delta_{i,t}} := \chi_{\Delta_{i,t}}(\hat{\mathbf{X}}) = \int_{\Delta_{i,t}} |x\rangle\langle x| dx$, i.e. the spectral projector of $\hat{\mathbf{X}}$ projecting onto the subspace corresponding to the set $\Delta_{i,t}$. The trace norms $\left\| \hat{\mathbf{P}}_{\Delta_i} \mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0}) \hat{\mathbf{P}}_{\Delta_j} \right\|_1$ are in general quite difficult to estimate. We present below two approaches; one is an adaptation of the work in [46] and the other is an application of the main theorem of [50].

5.3.1 Bounds of the Kupsch Kind [46]

One approach to estimating the trace norms in the inequality (5.43) below invokes some ideas from Kupsch's seminal paper on decoherence [46], where it is proven that

$$\| P_{\Delta_i} \mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0}) P_{\Delta_j} \| \leq C(1 + \delta^2 \psi(t))^{-\gamma} \tag{5.43}$$

for intervals Δ_j and Δ_i separated by a distance $\delta > 0$. Where $\psi(t) \geq 0$ is a function that diverges for $t \to \infty$, γ an exponent which can be large, and C is some constant; unfortunately there is no proof of this claim present in [46], the author is therefore led to believe that there is perhaps something to do with the Paley -Wiener theorem [3] working in the background, or maybe some basic Harmonic Analysis. Given that we will not be using (5.43), but a variant rather, we will not worry too much about deriving (5.43) ourselves.

We now present the variant to the bound found in the appendix of [46]. Again, we will focus on the case where $\hat{\mathbf{X}}$ is the position operator.

Theorem 5.3.1 (Adapting Kupsch's Bounds [46])

Let us fix $t > 0$ and let $\hat{\boldsymbol{\rho}}_t$ be some density operator which may be represented, using the generalized eigenvectors of the position operator $\hat{\mathbf{X}}$, as

$$\hat{\boldsymbol{\rho}}_t = \int\int \Gamma(t,x,y)K(x,y)\big|x\big\rangle\big\langle y\big|dxdy \tag{5.44}$$

where $\Gamma(t,x,y)$ and $\hat{\mathbf{P}}_{\Delta_{i,t}}$ are defined here as they were in (5.12); $K(x,y)$ is the kernel of $\hat{\boldsymbol{\rho}}_0$. Then,

$$\|\hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_{j,t}}\|_1 \leq \sup_{(x,y)\in\Delta_{i,t}\times\Delta_{j,t}} \Big(2|\Gamma(t,x,y)| + |\Delta_{j,t}||\partial_y\Gamma(t,x,y)|\Big) \tag{5.45}$$

when

$$\Big|\Delta_{i,t}\times\Delta_{j,t}\cap \mathbf{supp}\big\{\Gamma(t,x,y)K(x,y)\big\}\Big| \neq 0 \tag{5.46}$$

otherwise

$$\|\hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_{j,t}}\|_1 = 0 \tag{5.47}$$

Proof. CASE 1)

If $\Delta_{i,t}\times\Delta_{j,t}$ is such that

$$\Big|\Delta_{i,t}\times\Delta_{j,t}\cap \mathbf{supp}\big\{\Gamma(t,x,y)K(x,y)\big\}\Big| = 0 \tag{5.48}$$

then

$$\Big\|\hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_{j,t}}\Big\|_1 = \Big\|\int_{\Delta_{i,t}}\int_{\Delta_{j,t}} \Gamma(t,x,y)K(x,y)\big|x\big\rangle\big\langle y\big|dxdy\Big\|_1 = \tag{5.49}$$

$$\Big\|\int_{\Delta_{i,t}}\int_{\Delta_{j,t}} 0\big|x\big\rangle\big\langle y\big|dxdy\Big\|_1 = 0 \tag{5.50}$$

CASE 2)

Now we assume that

$$\Big|\Delta_{i,t}\times\Delta_{j,t}\cap \mathbf{supp}\big\{\Gamma(t,x,y)K(x,y)\big\}\Big| \neq 0 \tag{5.51}$$

Let us begin by considering the operator

$$\hat{\mathbf{T}}_t(y) := \int_{\Delta_{i,t}} \Gamma(t,x,y)\big|x\big\rangle\big\langle x\big|dx. \tag{5.52}$$

Where i is fixed. $\hat{\mathbf{T}}_t(y)$ is a differentiable family of operators, with respect to y, with the operator norm estimate

$$\big\|\hat{\mathbf{T}}_t(y)\big\| \leq \sup_{x\in\Delta_{i,t}} |\Gamma(t,x,y)| \tag{5.53}$$

The latter follows from the calculation below.

$$\left\|\hat{\mathbf{T}}_t(y)\right\|^2 = \sup_{\||\psi\rangle\|=1} \left\|\hat{\mathbf{T}}_t(y)|\psi\rangle\right\|^2 = \tag{5.54}$$

$$\sup_{\||\psi\rangle\|=1} \int_{\Delta_{i,t}} \int_{\Delta_{i,t}} \Gamma(t,x',y)^* \Gamma(t,x,y) \langle\psi|x'\rangle\langle x'|x\rangle\langle x|\psi\rangle dx' dx = \tag{5.55}$$

$$\sup_{\||\psi\rangle\|=1} \int_{\Delta_{i,t}} |\Gamma(t,x,y)|^2 \langle\psi|x\rangle\langle x|\psi\rangle dx \leq \tag{5.56}$$

$$\sup_{x\in\Delta_{i,t}} |\Gamma(t,x,y)|^2 \sup_{\||\psi\rangle\|=1} \int_{\Delta_{i,t}} |\psi(x)|^2 dx \leq \sup_{x\in\Delta_{i,t}} |\Gamma(t,x,y)|^2 \tag{5.57}$$

In a similar way, we may bound the operator $\hat{\mathbf{T}}'_t(y) := \int_{\Delta_{i,t}} \Gamma^{'}(t,x,y)|x\rangle\langle x|dx$. Where $\Gamma^{'}(t,x,y) :=$

$\partial_y \Gamma(t,x,y)$. i.e.

$$\left\|\hat{\mathbf{T}}'_t(y)\right\| \leq \sup_{x\in\Delta_{i,t}} |\Gamma^{'}(t,x,y)| \tag{5.58}$$

Furthermore, define $\hat{\mathbf{J}}_t(y) := \hat{\mathbf{T}}_t(y)\hat{\boldsymbol{\rho}}_0$ and $\hat{\mathbf{J}}^{'}_t(y) := \hat{\mathbf{T}}^{'}_t(y)\hat{\boldsymbol{\rho}}_0$. These operators also have uniform estimates; using the estimates computed above, and the inequality $\|\hat{\mathbf{A}}\hat{\mathbf{C}}\|_1 \leq \|\hat{\mathbf{A}}\|\|\hat{\mathbf{C}}\|_1$ one may easily show that

$$\left\|\hat{\mathbf{J}}_t(y)\right\|_1 \leq \sup_{x\in\Delta_{i,t}} |\Gamma(t,x,y)| \left\|\hat{\boldsymbol{\rho}}_0\right\|_1 = \sup_{x\in\Delta_{i,t}} |\Gamma(t,x,y)| \tag{5.59}$$

and that

$$\left\|\hat{\mathbf{J}}^{'}_t(y)\right\|_1 \leq \sup_{x\in\Delta_{i,t}} |\Gamma^{'}(t,x,y)| \left\|\hat{\boldsymbol{\rho}}_0\right\|_1 = \sup_{x\in\Delta_{i,t}} |\Gamma^{'}(t,x,y)|. \tag{5.60}$$

We now clarify the relationship between the operator $\hat{\mathbf{T}}'_t(y)$ and the weak derivative $\partial_y\langle\psi|\hat{\mathbf{T}}_t(y)|\phi\rangle$.

$$\partial_y \langle\psi|\hat{\mathbf{T}}_t(y)|\phi\rangle = \partial_y \int_{\Delta_{i,t}} \Gamma(t,x,y)\langle\psi|x\rangle\langle x|\phi\rangle dx \tag{5.61}$$

Assuming that $\Gamma(t,x,y)$ is $C^1(\Delta_{i,t})$ with respect to y we may now swap the order of the integral and the derivative.

$$\partial_y \int_{\Delta_{i,t}} \Gamma(t,x,y)\langle\psi|x\rangle\langle x|\phi\rangle dx = \int_{\Delta_{i,t}} \partial_y \Gamma(t,x,y)\langle\psi|x\rangle\langle x|\phi\rangle dx = \tag{5.62}$$

$$\int_{\Delta_{i,t}} \Gamma^{'}(t,x,y)\langle\psi|x\rangle\langle x|\phi\rangle dx = \langle\psi|\Big(\int_{\Delta_{i,t}} \Gamma^{'}(t,x,y)|x\rangle\langle x|dx\Big)|\phi\rangle = \langle\psi|\hat{\boldsymbol{T}}^{'}_t(y)|\phi\rangle \tag{5.63}$$

We therefore have

$$\partial_y \langle\psi|\hat{\mathbf{T}}_t(y)|\phi\rangle = \langle\psi|\hat{\mathbf{T}}'_t(y)|\phi\rangle \tag{5.64}$$

Now, for all intervals $\Delta_{j,t}$, we have $\int_{\Delta_{j,t}} \hat{\mathbf{J}}_t(y)|y\rangle\langle y|dy = \hat{\mathbf{P}}_{\Delta_{i,t}}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_{j,t}}$. Let us define $\Delta_{j,t} :=$

$[a_j(t), b_j(t)]$. We will show that the following identity holds.

$$\int_{\Delta_{j,t}} \hat{\mathbf{J}}_t(y)\big|y\big\rangle\big\langle y\big|dy = \hat{\mathbf{J}}_t(b_j(t))\hat{\mathbf{P}}_{(-\infty,b_j(t)]} - \hat{\mathbf{J}}_t(a_j(t))\hat{\mathbf{P}}_{(-\infty,a_j(t)]} - \int_{\Delta_{j,t}} \hat{\mathbf{J}}'_t(y)\hat{\mathbf{P}}_{(-\infty,y]}dy. \tag{5.65}$$

In what follows, we will use the purity of $\hat{\boldsymbol{\rho}}_0$ and write it in bra-ket notation, i.e. let $\big|\xi_0\big\rangle\big\langle\xi_0\big| := \hat{\boldsymbol{\rho}}_0$. Now, For arbitrary $\big|\psi\big\rangle$ and $\big|\phi\big\rangle$

$$\big\langle\psi\big|\int_{\Delta_{j,t}} \hat{\mathbf{J}}_t(y)\big|y\big\rangle\big\langle y\big|dy\big|\phi\big\rangle = \int_{\Delta_{j,t}} \big\langle\psi\big|\hat{\mathbf{J}}_t(y)\big|y\big\rangle\big\langle y\big|\phi\big\rangle dy. \tag{5.66}$$

By the definition of $\hat{\mathbf{J}}_t(y)$ one has

$$\big\langle\psi\big|\hat{\mathbf{J}}_t(y)\big|y\big\rangle = \big\langle\psi\big|\hat{\mathbf{T}}_t(y)\hat{\boldsymbol{\rho}}_0\big|y\big\rangle = \big\langle\psi\big|\hat{\mathbf{T}}_t(y)\big|\xi_0\big\rangle\big\langle\xi_0\big|y\big\rangle. \tag{5.67}$$

Picking up from (5.66).

$$(5.66) = \int_{\Delta_{j,t}} \big\langle\psi\big|\hat{\mathbf{T}}_t(y)\big|\xi_0\big\rangle\big\langle\xi_0\big|y\big\rangle\big\langle y\big|\phi\big\rangle dy = \tag{5.68}$$

$$\left[\big\langle\psi\big|\hat{\mathbf{T}}_t(y)\big|\xi_0\big\rangle\big\langle\xi_0\big|\left(\int_{-\infty}^{y} dy'\big|y'\big\rangle\big\langle y'\big|\right)\big|\phi\big\rangle\right]\Bigg|_{a_j(t)}^{b_j(t)} - \int_{\Delta_{j,t}}\left(\int_{-\infty}^{y}\big\langle\xi_0\big|y'\big\rangle\big\langle y'\big|\phi\big\rangle dy'\right)d\left(\big\langle\psi\big|\hat{\mathbf{T}}_t(y)\big|\xi_0\big\rangle\right) = \tag{5.69}$$

$$\left[\big\langle\psi\big|\hat{\mathbf{T}}_t(y)\big|\xi_0\big\rangle\big\langle\xi_0\big|P_{(-\infty,y]}\big|\phi\big\rangle\right]\Bigg|_{a_j(t)}^{b_j(t)} - \int_{\Delta_{j,t}}\left(\int_{-\infty}^{y}\big\langle\xi_0\big|y'\big\rangle\big\langle y'\big|\phi\big\rangle dy'\right)\left(\big\langle\psi\big|\hat{\mathbf{T}}_t(y)\big|\xi_0\big\rangle\right)' dy = \tag{5.70}$$

$$\left[\big\langle\psi\big|\hat{\mathbf{T}}_t(y)\big|\xi_0\big\rangle\big\langle\xi_0\big|\hat{\mathbf{P}}_{(-\infty,y]}\big|\phi\big\rangle\right]\Bigg|_{a_j(t)}^{b_j(t)} - \int_{\Delta_{j,t}}\left(\big\langle\psi\big|\hat{\mathbf{T}}_t(y)\big|\xi_0\big\rangle\right)'\big\langle\xi_0\big|\hat{\mathbf{P}}_{(-\infty,y]}\big|\phi\big\rangle dy = \tag{5.71}$$

$$\big\langle\psi\big|\left[\hat{\mathbf{T}}_t(y)\big|\xi_0\big\rangle\big\langle\xi_0\big|\hat{\mathbf{P}}_{(-\infty,y]}\right]\Bigg|_{a_j(t)}^{b_j(t)}\big|\phi\big\rangle - \big\langle\psi\big|\left(\int_{\Delta_{j,t}}\hat{\mathbf{T}}'_t(y)\big|\xi_0\big\rangle\big\langle\xi_0\big|\hat{\mathbf{P}}_{(-\infty,y]}dy\right)\big|\phi\big\rangle = \tag{5.72}$$

$$\big\langle\psi\big|\left(\hat{\mathbf{J}}_t(y)\hat{\mathbf{P}}_{(-\infty,y]}\Bigg|_{a_j(t)}^{b_j(t)} - \int_{\Delta_{j,t}}\hat{\mathbf{J}}'_t(y)\hat{\mathbf{P}}_{(-\infty,y]}dy\right)\big|\phi\big\rangle = \tag{5.73}$$

$$\big\langle\psi\big|\left(\hat{\mathbf{J}}_t(b_j(t))\hat{\mathbf{P}}_{(-\infty,b_j(t)]} - \hat{J}_t(a_j(t))\hat{\mathbf{P}}_{(-\infty,a_j(t)]} - \int_{\Delta_{j,t}}\hat{\mathbf{J}}'_t(y)\hat{\mathbf{P}}_{(-\infty,y]}dy\right)\big|\phi\big\rangle \tag{5.74}$$

and so

$$\int_{\Delta_{j,t}} \hat{\mathbf{J}}_t(y)\big|y\big\rangle\big\langle y\big|dy = \hat{\mathbf{J}}_t(b_j(t))\hat{\mathbf{P}}_{(-\infty,b_j(t)]} - \hat{\mathbf{J}}_t(a_j(t))\hat{\mathbf{P}}_{(-\infty,a_j(t)]} - \int_{\Delta_{j,t}} \hat{\mathbf{J}}'_t(y)\hat{\mathbf{P}}_{(-\infty,y]}dy. \tag{5.75}$$

Consequently

$$\big\|\hat{\mathbf{P}}_{\Delta_{i.t}}\hat{\boldsymbol{\rho}}_t\hat{\mathbf{P}}_{\Delta_{j,t}}\big\|_1 = \left\|\int_{\Delta_{j,t}}\hat{\mathbf{J}}_t(y)|y\rangle\langle y|dy\right\|_1 = \tag{5.76}$$

$$\left\|\hat{\mathbf{J}}_t(b_j(t))\hat{\mathbf{P}}_{(-\infty,b_j(t)]} - \hat{\mathbf{J}}_t(a_j(t))\hat{\mathbf{P}}_{(-\infty,a_j(t)]} - \int_{\Delta_{j,t}} \hat{\mathbf{J}}'_t(y)\hat{\mathbf{P}}_{(-\infty,y]}dy\right\|_1 \leq \tag{5.77}$$

$$\left\|\hat{\mathbf{J}}_t(b_j(t))\hat{\mathbf{P}}_{(-\infty,b_j(t)]}\right\|_1 + \left\|\hat{\mathbf{J}}_t(a_j(t))\hat{\mathbf{P}}_{(-\infty,a_j(t)]}\right\|_1 + \left\|\int_{\Delta_{j,t}} \hat{\mathbf{J}}'_t(y)\hat{\mathbf{P}}_{(-\infty,y]}dy\right\|_1 \leq \tag{5.78}$$

$$\left\|\hat{\mathbf{J}}_t(b_j(t))\right\|_1\left\|\hat{\mathbf{P}}_{(-\infty,b_j(t)]}\right\| + \left\|\hat{\mathbf{J}}_t(a_j(t))\right\|_1\left\|\hat{\mathbf{P}}_{(-\infty,a_j(t)]}\right\| + \int_{\Delta_{j,t}} \left\|\hat{\mathbf{J}}_t(y)'\hat{\mathbf{P}}_{(-\infty,y]}\right\|_1 dy \leq \tag{5.79}$$

$$\left\|\hat{\mathbf{J}}_t(b_j(t))\right\|_1 + \left\|\hat{\mathbf{J}}_t(a_j(t))\right\|_1 + \int_{\Delta_{j,t}} \left\|\hat{\mathbf{J}}'_t(y)\right\|_1\left\|\hat{\mathbf{P}}_{(-\infty,y]}\right\|dy = \tag{5.80}$$

$$\left\|\hat{\mathbf{J}}_t(b_j(t))\right\|_1 + \left\|\hat{\mathbf{J}}_t(a_j(t))\right\|_1 + \int_{\Delta_{j,t}} \left\|\hat{\mathbf{J}}'_t(y)\right\|_1 dy \leq \tag{5.81}$$

$$\left\|\hat{\mathbf{J}}_t(b_j(t))\right\|_1 + \left\|\hat{\mathbf{J}}_t(a_j(t))\right\|_1 + |\Delta_{j,t}| \sup_{y\in\Delta_{j,t}} \left\|\hat{\mathbf{J}}'_t(y)\right\|_1 dy \leq \tag{5.82}$$

$$\sup_{x\in\Delta_{i,t}} |\Gamma(t,x,b_j(t))| + \sup_{x\in\Delta_{i,t}} |\Gamma(t,x,a_j(t))| + |\Delta_{j,t}| \sup_{\substack{x\in\Delta_{i,t}\\ y\in\Delta_{j,t}}} |\partial_y\Gamma(t,x,y)| \leq \tag{5.83}$$

$$\sup_{\substack{(x,y)\in\\ \Delta_{i,t}\times\Delta_{j,t}}} \left(2|\Gamma(t,x,y)| + |\Delta_{j,t}\Gamma^{'}(t,x,y)|\right) \tag{5.84}$$

□

5.3.2 Another Way to Estimate the Off-diagonal Terms (5.26)

In the previous section, the kernel term $\Gamma(t,x,y)$ characterizing the non-unitarity evolution of the density operator in the hypothesis of Theorem 5.3.1 was treated in rather general terms. However, if more is known about the kernel $\Gamma(t,x,y)$, then one may employ yet another technique for the estimation of $\left\|\hat{\mathbf{P}}_{\Delta_{i,t}}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_{j,t}}\right\|_1$. Namely, we will utilize the following theorem from [50] in order to bound the non-diagonal terms for the case where it is known how to express $\hat{\mathbf{P}}_{\Delta_{i,t}}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_{j,t}}$ as a product of two trace class operators acting in some measurable $L^2(M,\mu)$. We will focus on the case where $M = \mathbb{R}$ and μ is just the Lebesgue measure.

THEOREM 5.3.2 (THEOREM FROM STOLZ ET AL [50])

Let $A(x,z)$ and $B(z,x) \in L^2(\mathbb{R})\,\forall\, z \in \mathbb{R}$ and

$$\int \|A(\cdot,z)\|_{L^2(\mathbb{R})}\|B(z,\cdot)\|_{L^2(\mathbb{R})}dz < \infty. \tag{5.85}$$

Then there is a trace class operator $\hat{\mathbf{A}}\hat{\mathbf{B}}$ acting in $L^2(\mathbb{R})$ with kernel

$$AB(x,y) = \int A(x,z)B(z,y)dz \tag{5.86}$$

such that

$$\left\|\hat{\mathbf{A}}\hat{\mathbf{B}}\right\|_1 \leq \int \|A(\cdot,z)\|_{L^2(\mathbb{R})}\|B(z,\cdot)\|_{L^2(\mathbb{R})}dz \tag{5.87}$$

The kernel operators $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ also act in $L^2(\mathbb{R})$ and have respective kernels $A(x,y)$ and $B(x,y)$.

The first step toward the employment of Theorem 5.3.2 to the estimation of the terms (5.42) is rewriting $\hat{\mathbf{P}}_{\Delta_{i,t}}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_{j,t}}$ as a product of of two operators. Let us fix t, and note that the main challenge is the kernel $\Gamma(t,x,y)$. Expanding, we see that

$$\hat{\mathbf{P}}_{\Delta_{i,t}}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_{j,t}} = \int\int \Gamma(t,x,y)\psi_{S_0}(x)\psi^*_{S_0}(y)\chi_{\Delta_{i,t}}(x)\chi_{\Delta_{j,t}}(y)\big|x\big\rangle\big\langle y\big|dxdy. \tag{5.88}$$

Where $\psi^*_{S_0}(x)\psi_{S_0}(y) = K(x,y)$ is the kernel of $\hat{\boldsymbol{\rho}}_{S_0}$. If the kernel $\Gamma(t,x,y)$ were to have a decomposition of the form

$$\Gamma(t,x,y) = \int \phi(t,x,z)\eta(t,y,z)dz \tag{5.89}$$

with

$$\int |\eta(t,x,z)\phi(t,y,z)|^2dz < \infty \tag{5.90}$$

for all $(t,x,y) \in \mathbb{R}^2$, then Theorem 5.3.2 would be applicable for any t. Assuming (5.90) we have

$$\left\|\hat{\mathbf{P}}_{\Delta_{i,t}}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_{j,t}}\right\|_1 \leq \tag{5.91}$$

$$\int\left(\left[\int |\phi(t,x,z)\psi_{S_0}(x)\chi_{\Delta_{i,t}}(x)|^2dx\right]\left[\int |\eta(t,y,z)\psi_{S_0}(y)\chi_{\Delta_{j,t}}(y)|^2dy\right]\right)dz = \tag{5.92}$$

$$\int\int\left(\left[\int |\phi(t,x,z)\eta(t,y,z)|^2dz\right]|\psi_{S_0}(x)\chi_{\Delta_{i,t}}(x)|^2|\psi_{S_0}(y)\chi_{\Delta_{j,t}}(y)|^2\right)dxdy \leq \tag{5.93}$$

$$\left[\max_{(x,y)\in\mathbb{R}^2}\int |\phi(t,x,z)\eta(t,y,z)|^2dz\right]\int\int\left(|\psi_{S_0}(x)\chi_{\Delta_{i,t}}(x)|^2|\psi_{S_0}(y)\chi_{\Delta_{j,t}}(y)|^2\right)dxdy = \tag{5.94}$$

$$\left[\max_{(x,y)\mathbb{R}^2}\int |\phi(t,x,z)\eta(t,y,z)|^2dz\right]\Big(\int_{\Delta_{i,t}} |\psi_{S_0}(x)|^2dx\Big)\Big(\int_{\Delta_{j,t}} |\psi_{S_0}(y)|^2dy\Big) \leq \tag{5.95}$$

$$\max_{(x,y)\in\mathbb{R}^2}\int|\phi(t,x,z)\eta(t,y,z)|^2dz<\infty \tag{5.96}$$

The Hypothesis of Theorem 5.3.2 is therefore satisfied, for all $t>0$, and we conclude that

$$\left\|\hat{\mathbf{P}}_{\Delta_{i,t}}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_{j,t}}\right\|_1\leq\int\|A_{i,t}(\cdot,z)\|_{L^2(\mathbb{R})}\|B_{j,t}(z,\cdot)\|_{L^2(\mathbb{R})}dz \tag{5.97}$$

with $A_{i,t}(x,z):=\phi(t,x,z)\psi_{S_0}(x)\chi_{\Delta_{i,t}}(x)$ and $B_{j,t}(z,x):=\eta(t,x,z)\psi^*_{S_0}(x)\chi_{\Delta_{j,t}}(x)$.

We now formalize the above as a corollary to Theorem 5.3.2.

Corollary 5.3.1 (Corollary to Theorem 5.3.2)

If the kernel $\Gamma(t,x,y)$ has a decomposition

$$\Gamma(t,x,y)=\int\phi(t,x,z)\eta(t,y,z)dz \tag{5.98}$$

with

$$\int|\eta(t,x,z)\phi(t,y,z)|^2dz<\infty \tag{5.99}$$

for all $(t,x,y)\in[0,\infty)\times\mathbb{R}^2$, then

$$\left\|\hat{\mathbf{P}}_{\Delta_{i,t}}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_{j,t}}\right\|_1\leq\int\|A_{i,t}(\cdot,z)\|_{L^2(\mathbb{R})}\|B_{j,t}(z,\cdot)\|_{L^2(\mathbb{R})}dz \tag{5.100}$$

with $A_{i,t}(x,z):=\phi(t,x,z)\psi_{S_0}(x)\chi_{\Delta_{i,t}}(x)$ and $B_{j,t}(z,x):=\eta(t,x,z)\psi^*_{S_0}(x)\chi_{\Delta_{j,t}}(x)$.

Proof. The proof is in the preceding discussion concluding with equation (5.97). □

Example

For the case where the kernel $\Gamma(t,x,y)=e^{-t^n\alpha(x-y)^2}$, $n>0,t>0$, we may express such a function as a convolution of Gaussians. Namely,

$$\Gamma(t,x,y)=e^{-t^n\alpha(x-y)^2}=2\sqrt{\frac{t^n\alpha}{\pi}}\int e^{-2t^n\alpha(x-z)^2}e^{-2t^n\alpha(y-z)^2}dz. \tag{5.101}$$

In this case the ϕ and η from (5.90) are just

$$\phi(t,x,z)=\sqrt{2\sqrt{\frac{t^n\alpha}{\pi}}}e^{-2t^n\alpha(x-z)^2} \tag{5.102}$$

and

$$\eta(t,y,z)=\sqrt{2\sqrt{\frac{t^n\alpha}{\pi}}}e^{-2t^n\alpha(y-z)^2}. \tag{5.103}$$

Notice that here

$$\int |\phi(t,x,z)\eta(t,y,z)|^2 dz = \frac{4t^n}{\pi}\int e^{-4t^n\alpha(x-z)^2}e^{-4t^n\alpha(y-z)^2}dz = \frac{4t^n\alpha}{\pi}\frac{\sqrt{\pi}}{2^{\frac{3}{2}}\sqrt{t^n\alpha}}e^{-2t^n\alpha(x-y)^2} = \tag{5.104}$$

$$\frac{\sqrt{2t^n\alpha}}{\sqrt{\pi}}e^{-2t^n\alpha(x-y)^2} < \infty \tag{5.105}$$

for all $t > 0$ and $(x, y) \in \mathbb{R}^2$. With such a ϕ and η, we can bound the corresponding inequality (5.97) as follows.

$$\left\|\hat{\mathbf{P}}_{\Delta_{i,t}}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_{j,t}}\right\|_1 \leq \tag{5.106}$$

$$\int\int\left(\left[\int |\phi(t,x,z)\eta(t,y,z)|^2 dz\right]|\psi_{S_0}(x)\chi_{\Delta_{i,t}}(x)|^2|\psi_{S_0}(y)\chi_{\Delta_{j,t}}(y)|^2\right)dxdy = \tag{5.107}$$

$$\int\int\left(\frac{\sqrt{2t^n\alpha}}{\sqrt{\pi}}e^{-2t^n\alpha(x-y)^2}|\psi_{S_0}(x)\chi_{\Delta_{i,t}}(x)|^2|\psi_{S_0}(y)\chi_{\Delta_{j,t}}(y)|^2\right)dxdy = \tag{5.108}$$

$$\int_{\Delta_{i,t}}\int_{\Delta_{j,t}}\left(\frac{\sqrt{2t^n\alpha}}{\sqrt{\pi}}e^{-2t^n\alpha(x-y)^2}|\psi_{S_0}(x)|^2|\psi_{S_0}(y)|^2\right)dxdy \tag{5.109}$$

The Kernel $\frac{\sqrt{2t^n\alpha}}{\sqrt{\pi}}e^{-2t^n\alpha(x-y)^2}$ is a delta sequence with respect to t. Therefore, as t becomes arbitrarily large, the support of the integrand of (5.109) narrows along the diagonal elements $x \neq y$ of $\Delta_{i,t} \times \Delta_{j,t}$. Hence, whenever $x \in \Delta_{i,t}$, $y \in \Delta_{j,t}$, and $(\Delta_{i,t} \cap \Delta_{j,t} = \emptyset)$, (5.109) vanishes as $t \to \infty$.

Gaussian states are popular enough that the latter discussion merits emphasis as yet another corollary of Theorem 5.3.2.

Corollary 5.3.2 (Theorem (5.3.2) with Gaussian assumptions for $\Gamma(t, x, y)$)

Fix $t > 0$. Now let

$$\Gamma(t,x,y) = e^{-t^n\alpha(x-y)^2} = 2\sqrt{\frac{t^n\alpha}{\pi}}\int e^{-2t^n\alpha(x-z)^2}e^{-2t^n\alpha(y-z)^2}dz \tag{5.110}$$

where $n > 0$, and assume that $\Delta_{i,t} \cap \Delta_{j,t} = \emptyset$, then

$$1)\ \left\|\hat{\mathbf{P}}_{\Delta_{i,t}}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_{j,t}}\right\|_1 \leq \int_{\Delta_{i,t}}\int_{\Delta_{j,t}}\left(\frac{\sqrt{2t^n\alpha}}{\sqrt{\pi}}e^{-2t^n\alpha(x-y)^2}|\psi_{S_0}(x)|^2|\psi_{S_0}(y)|^2\right)dxdy \tag{5.111}$$

S

$$2)\ \lim_{t\to\infty}\left\|\hat{\mathbf{P}}_{\Delta_{i,t}}\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_0})\hat{\mathbf{P}}_{\Delta_{j,t}}\right\|_1 = 0 \tag{5.112}$$

Proof. The proof is in the preceding discussion concluding with equation (5.109). □

5.4 Estimating the Diagonal Terms (5.25)

We have hitherto developed the tools necessary to estimate the trace norm of the off-diagonal terms (5.26) arising from estimates of the optimization problem (5.20). To conclude our estimation of the optimization problem (5.20) we now study the diagonal terms (5.25). i.e.

$$\min_{PVM} \left\| \sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right) - \frac{1}{\mathscr{N}(t)} \sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\right) \right\|_1 \tag{5.113}$$

The minimization is taken over all PVMs resolving the identity operator of the space associated with the environmental degrees of freedom. Recall that $\sum_i \sum_j \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{j,t}} \otimes \mathbb{I}\right)$ is just another way of writing $\hat{\boldsymbol{\rho}}_t$ (see 5.12). The use of the PVM $\left\{\hat{\mathbf{P}}_{\Delta_{i,t}}\right\}_i$ in the first term of the difference in (5.113) is just technical. However, the usage of the same PVM on the second term in the difference of (5.113) does imply measurement of the von Neumann type performed on the system; i.e in a local sense in the sense Definition 2.2.3. When estimating the off-diagonal terms (5.26), we were only tasked with studying its asymptotic behavior with respect to t since the families of PVM acting on the system's degree of freedom, $\hat{\mathbf{P}}_{\Delta_{i,t}}$ were assumed to be predetermined. For the case of the diagonal terms (5.25), we are now tasked with studying the limit as t goes to infinity of a term which depends on a minimization optimization, namely (5.113). This is now a more challenging problem since a way to estimate the optimal PVM acting on the environmental degrees of freedom in (5.113) for each value of $t > 0$ is needed to understand its asymptotic behavior.

One might have already noted that the map $\sum_i \hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}(...)\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}$ is unlike the related measurements of von Neumann type seen in Definition 2.2.3 since they do not preserve the trace. Both are indeed completely positive maps, but the latter map turns out to reduce the trace in general, i.e.

$$Tr\left\{ \sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\right) \right\} \leq Tr\{\hat{\boldsymbol{\rho}}_t\} = 1. \tag{5.114}$$

Indeed, the PVM $\{\hat{\mathbf{P}}_{\Delta_i} \otimes \hat{\mathbf{P}}_i^{E_t}\}_i$ by itself does not describe a measurement for the product of the system's and environment's Hilbert spaces because it does not resolve the identity operator acting over the entire Hilbert space $\mathscr{H}_S \otimes \mathscr{H}_E$ but rather the identity of a subspace of $\mathscr{H}_S \otimes \mathscr{H}_E$; hence the need for the normalization constant $\mathscr{N}(t)$ in (5.113). The associated PVM, which preserves trace, and resolves the identity of $\mathscr{H}_S \otimes \mathscr{H}_E$, is indeed the family of projectors $\left\{\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_j^{E_t}\right\}_{i,j}$.This set includes outcomes pertaining to the case where the environment E is measured to be in the state labeled by the index j which differs from the outcome measured by the system S $i \neq j$. Hence, we exclude the $i \neq j$ terms when constructing the approximating SBS state (5.18).

Let us now estimate (5.113). We begin by rewriting the operator $\sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right)$ in the form described by the following Lemma:

Lemma 5.4.1 (Rewriting (5.113))

$$\sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right) = \sum_i \bar{p}_i(t)\mathscr{U}_t\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\big) \otimes \hat{\boldsymbol{\rho}}^{E_0}\big) \tag{5.115}$$

where

$$\hat{\boldsymbol{\rho}}_{S_{i,t}} = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{i,t}(x,y)|x\rangle\langle y|dxdy \tag{5.116}$$

$$K_{i,t}(x,y) := \frac{\mathbb{1}_{\Delta_{i,t}}(x)\psi(x)}{\sqrt{\bar{p}_i(t)}}\frac{\mathbb{1}_{\Delta_{i,t}}(y)\psi^*(y)}{\sqrt{\bar{p}_i(t)}} \tag{5.117}$$

$$\bar{p}_i(t) := \int_{\Delta_{i,t}} K(x,x)dx, \tag{5.118}$$

$$\psi(x)\psi^*(y) = K(x,y) \tag{5.119}$$

and recalling that

$$\mathscr{U}_t(\hat{\mathbf{A}}) := e^{-it\gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}}\hat{\mathbf{A}}e^{it\gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}} \tag{5.120}$$

Proof.

$$\sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right) = \sum_i \int_{\Delta_{i,t}}\int_{\Delta_{i,t}} K(x,y)\Gamma(t,x,y)\big|x\big\rangle\big\langle y\big| \otimes \hat{\boldsymbol{\rho}}^{E_t}_{x,y}dxdy = \tag{5.121}$$

$$\sum_i \bar{p}_i(t)\int_{\Delta_{i,t}}\int_{\Delta_{i,t}} \frac{K(x,y)}{\bar{p}_i(t)}\Gamma(t,x,y)|x\rangle\langle y| \otimes \hat{\boldsymbol{\rho}}^{E_t}_{x,y}dxdy = \tag{5.122}$$

where

$$\bar{p}_i(t) := \int_{\Delta_{i,t}} K(x,x)dx. \tag{5.123}$$

That is,

$$(5.122) = \sum_i \bar{p}_i(t)\int_{\mathbb{R}}\int_{\mathbb{R}} K_{i,t}(x,y)\Gamma(t,x,y)|x\rangle\langle y| \otimes \hat{\boldsymbol{\rho}}^{E_t}_{x,y}dxdy \tag{5.124}$$

recalling that $K(x,y) = \psi(x)\psi^*(y)$ owing to the purity of $\hat{\boldsymbol{\rho}}_{S_0}$, we define

$$K_{i,t}(x,y) := \mathbb{1}_{\Delta_{i,t}}(x)\mathbb{1}_{\Delta_{i,t}}(y)\frac{K(x,y)}{\bar{p}_i(t)} = \frac{\mathbb{1}_{\Delta_{i,t}}(x)\psi(x)}{\sqrt{\bar{p}_i(t)}}\frac{\mathbb{1}_{\Delta_{i,t}}(y)\psi^*(y)}{\sqrt{\bar{p}_i(t)}}. \tag{5.125}$$

Furthermore, let us define

$$\psi_{S_{i,t}}(x) := \frac{\mathbb{1}_{\Delta_{i,t}}(x)\psi(x)}{\sqrt{\bar{p}_i(t)}} \tag{5.126}$$

and write

$$K_{i,t}(x,y) = \psi_{S_{i,t}}(x)\psi^*_{S_{i,t}}(y)\ \ \big(Kernel\ \ of\ \ \big|\psi_{S_{i,t}}\big\rangle\big\langle\psi_{S_{i,t}}\big|\ \ \big) \tag{5.127}$$

Finally,

$$(5.124) = \sum_i \bar{p}_i(t)\mathscr{U}_t\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\big) \otimes \hat{\boldsymbol{\rho}}^{E_0}\big) \tag{5.128}$$

where we define

$$\hat{\boldsymbol{\rho}}_{S_{i,t}} := \frac{\hat{\mathbf{P}}_{\Delta_{i,t}} \hat{\boldsymbol{\rho}}_{S_0} \hat{\mathbf{P}}_{\Delta_{i,t}}}{Tr\{\hat{\mathbf{P}}_{\Delta_{i,t}} \hat{\boldsymbol{\rho}}_{S_0} \hat{\mathbf{P}}_{\Delta_{i,t}}\}} = \int_{\mathbb{R}} \int_{\mathbb{R}} K_{i,t}(x,y) |x\rangle\langle y| dxdy \tag{5.129}$$

and utilize the definition of $\mathscr{U}_t$ presented in (5.6) with $g = \gamma$ and $N_E = 1$, i.e.

$$\mathscr{U}_t(\hat{\mathbf{A}}) := e^{-it\gamma \hat{\mathbf{X}} \otimes \hat{\mathbf{B}}} \hat{\mathbf{A}} e^{it\gamma \hat{\mathbf{X}} \otimes \hat{\mathbf{B}}}. \tag{5.130}$$

□

Employing Lemma 5.4.1 we may now write

$$\sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\right) = \sum_i \bar{p}_i(t) \left(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\right) \mathscr{U}_t\left(\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_{i,t}}) \otimes \hat{\boldsymbol{\rho}}^{E_0}\right) \left(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\right) \tag{5.131}$$

Finally, normalizing the operator (dividing by its trace) (5.131) we obtain an approximate SBSCV state to $\hat{\boldsymbol{\rho}}_t$.

$$\hat{\boldsymbol{\rho}}_{SBSCV,t} := \frac{1}{\mathscr{N}(t)} \sum_i \bar{p}_i(t) \left(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\right) \mathscr{U}_t\left(\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_{i,t}}) \otimes \hat{\boldsymbol{\rho}}^{E_0}\right) \left(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\right). \tag{5.132}$$

Just as we did for the case of discrete variables in the previous chapter (see equations (4.58) through (4.65)) we will be estimating $\left\|\hat{\boldsymbol{\rho}}_t - \mathscr{N}(t) \hat{\boldsymbol{\rho}}_{SBSCV,t}\right\|_1$ first, and then, using Lemma 4.4.1 we shall bound $\left\|\hat{\boldsymbol{\rho}}_t - \hat{\boldsymbol{\rho}}_{SBSCV,t}\right\|_1$.

The representation (5.132) makes transparent the structure of the dynamics being imposed on the total initial states $\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0}$ by making explicit all of the quantum maps generating the dynamics, i.e.

$$\Lambda_t \circ \mathscr{U}_t \circ (\mathscr{E}_t \otimes \mathcal{I}_E)\left(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0}\right) \tag{5.133}$$

where $\mathcal{I}_E$ is the identity map on the environmental degrees of freedom and

$$\Lambda_t(\hat{\mathbf{A}}) := \sum_i \frac{1}{\mathscr{N}(t)} \sum_i \bar{p}_i(t) \left(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\right) \hat{\mathbf{A}} \left(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\right). \tag{5.134}$$

It is clear that the quantum maps Λ_t and $\mathscr{E}_t \otimes \mathcal{I}_E$ commute due to their respective non-trivial influences taking effect only in complementary subspaces. What is more interesting and less obvious is the commutativity between $\mathscr{E}_t \otimes \mathcal{I}_E$ and the unitary map $\mathscr{U}_t$. Proving this is the content of the following lemma (Lemma 5.4.2); a lemma that we shall need for the proof of the main result of this chapter.

Lemma 5.4.2 (Commutativity of $\mathscr{E}_t \otimes \mathcal{I}_E$ and $\mathscr{U}_t$)

$$\mathscr{U}_t \circ (\mathscr{E}_t \otimes \mathcal{I}_E)\left(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0}\right) = (\mathscr{E}_t \otimes \mathcal{I}_E) \circ \mathscr{U}_t\left(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0}\right) \tag{5.135}$$

Proof. We remind the reader that we are always working within the framework discussed in the introduction to this chapter (see equations (5.3) through 5.11)). With this in mind, define $\mathscr{V}_t(\hat{\mathbf{A}}) :=$

$e^{-it\gamma'\hat{\mathbf{X}}\otimes\mathbb{I}_E\otimes\hat{\mathbf{B}}'}\big(\hat{\mathbf{A}}\big)e^{it\gamma'\hat{\mathbf{X}}\otimes\mathbb{I}_E\otimes\hat{\mathbf{B}}'}$. Then,

$$\mathscr{U}_t \circ \big(\mathscr{E}_t \otimes \mathcal{I}_E\big)\Big(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0}\Big) = \mathscr{U}_t\Bigg(Tr_{E'}\Big\{\mathscr{V}_t\Big(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0} \otimes \hat{\boldsymbol{\rho}}^{E_0'}\Big)\Big\}\Bigg) = \tag{5.136}$$

$$\Big(\mathscr{U}_t \otimes \mathcal{I}_{E'}\Big) \circ \Big(\mathcal{I}_{SE} \otimes Tr_{E'}\Big) \circ \mathscr{V}_t\Big(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0} \otimes \hat{\boldsymbol{\rho}}^{E_0'}\Big) \tag{5.137}$$

Where $\mathcal{I}_{SE}$ is the identity quantum map acting in the combined degrees of freedom of the system S and the environment E. By virtue of the fact that the quantum maps $\mathcal{I}_{SE} \otimes Tr_{E'}$ and $\mathscr{U}_t \otimes \mathcal{I}_{E'}$ produce non-trivial effects only on complementary subspaces, these two commute. Hence

$$(5.137) = \Big(\mathcal{I}_{SE} \otimes Tr_{E'}\Big) \circ \Big(\mathscr{U}_t \otimes \mathcal{I}_{E'}\Big) \circ \mathscr{V}_t\Big(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0} \otimes \hat{\boldsymbol{\rho}}^{E_0'}\Big) \tag{5.138}$$

Furthermore, it is easy to see that the generators of the unitary maps $\mathscr{U}_t \otimes \mathcal{I}_{E'}$ and $\mathscr{V}_t$ commute. Namely $\hat{\mathbf{X}} \otimes \hat{\mathbf{B}} \otimes \mathbb{I}_{E'}$ and $\hat{\mathbf{X}} \otimes \mathbb{I}_E \otimes \hat{\mathbf{B}}'$. We therefore have the following.

$$(5.138) = \Big(\mathcal{I}_{SE} \otimes Tr_{E'}\Big) \circ \mathscr{V}_t \circ \Big(\mathscr{U}_t \otimes \mathcal{I}_{E'}\Big)\Big(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0} \otimes \hat{\boldsymbol{\rho}}^{E_0'}\Big) = \tag{5.139}$$

$$\Big(\mathcal{I}_{SE} \otimes Tr_{E'}\Big) \circ \mathscr{V}_t\Big(\mathscr{U}_t\Big(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0}\Big) \otimes \hat{\boldsymbol{\rho}}^{E_0'}\Big) = \tag{5.140}$$

$$Tr_{E'}\Big\{\mathscr{V}_t\Big(\mathscr{U}_t\Big(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0}\Big) \otimes \hat{\boldsymbol{\rho}}^{E_0'}\Big)\Big\} = \big(\mathscr{E}_t \otimes \mathcal{I}_E\big)\Big(\mathscr{U}_t\Big(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0}\Big)\Big) = \tag{5.141}$$

$$\big(\mathscr{E}_t \otimes \mathcal{I}_E\big) \circ \mathscr{U}_t\Big(\hat{\boldsymbol{\rho}}_{S_0} \otimes \hat{\boldsymbol{\rho}}^{E_0}\Big) \tag{5.142}$$

□

Note that this proof is independent of the states $\hat{\boldsymbol{\rho}}_{S_0}$ and $\hat{\boldsymbol{\rho}}^{E_0}$.

The following corollary follows from Lemma 5.4.2.

Corollary 5.4.1 (A $\mathscr{E}_t$ independent estimate)

$$\Bigg\|\sum_i \Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\Big) - \sum_i \Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\Big)\Bigg\|_1 \leq \tag{5.143}$$

$$\sum_i \bar{p}_i(t)\Big\|\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}} \otimes \hat{\boldsymbol{\rho}}^{E_0}\big) - \big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}} \otimes \hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\big)\Big\|_1 \tag{5.144}$$

Proof.

$$\Bigg\|\sum_i \Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\Big) - \sum_i \Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\Big)\Bigg\|_1 = \tag{5.145}$$

$$\Bigg\|\sum_i \bar{p}_i(t)\Big(\mathscr{U}_t\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\big) \otimes \hat{\boldsymbol{\rho}}^{E_0}\big) - \big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\big) \otimes \hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\big)\Big)\Bigg\|_1 \leq \tag{5.146}$$

$$\sum_i \bar{p}_i \Big\| \mathscr{U}_t\big(\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_{i,t}}) \otimes \hat{\boldsymbol{\rho}}^{E_0}\big) - \big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\mathscr{E}_t(\hat{\boldsymbol{\rho}}_{S_{i,t}}) \otimes \hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\big) \Big\|_1 \leq \tag{5.147}$$

$$\sum_i \bar{p}_i \Big\| \mathscr{E}_t\Big(\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}} \otimes \hat{\boldsymbol{\rho}}^{E_0}\big) - \big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}} \otimes \hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\big) \Big) \Big\|_1 \leq \tag{5.148}$$

$$\sum_i \bar{p}_i \Big\| \mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}} \otimes \hat{\boldsymbol{\rho}}^{E_0}\big) - \big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}} \otimes \hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\big) \Big\|_1 \tag{5.149}$$

where we have used Theorem 5.4.2 going from (5.147) to (5.148), and in going from (5.148) to (5.149) we used the contractivity property of quantum maps (see Theorem 2.3.1). □

Without the effects of the quantum map $\mathscr{E}_t$, the term $\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}} \otimes \hat{\boldsymbol{\rho}}^{E_0}\big)$ in (5.144) is now pure; recall that $\hat{\boldsymbol{\rho}}^{E_0}$ was assumed pure and $\hat{\boldsymbol{\rho}}_{S_{i,t}} = \big|\psi_{S_{i,t}}\big\rangle\big\langle\psi_{S_{i,t}}\big|$ (5.127), where $\big|\psi_{S_i}\big\rangle$ is a pure state. To accentuate the latter we write $\hat{\boldsymbol{\rho}}_{S_{i,t}} = \big|\psi_{S_{i,t}}\big\rangle\big\langle\psi_{S_{i,t}}\big|$ in place of $\hat{\boldsymbol{\rho}}_{S_{i,t}}$, and we use the definition of the map $\mathscr{U}_t$ to express it as a left and right product of unitary operators.

$$\underline{\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}} \otimes \hat{\boldsymbol{\rho}}^{E_0}\big) = \hat{\mathbf{U}}_t\Big(\big|\psi_{S_{i,t}}\big\rangle\big\langle\psi_{S_{i,t}}\big| \otimes \big|\psi_{E_0}\big\rangle\big\langle\psi_{E_0}\big|\Big)\hat{\mathbf{U}}_t^\dagger} = \tag{5.150}$$

$$\underline{\bigg(\hat{\mathbf{U}}_t\Big(\big|\psi_{S_{i,t}}\big\rangle \otimes \big|\psi_{E_0}\big\rangle\Big)\bigg)\bigg(\hat{\mathbf{U}}_t\Big(\big|\psi_{S_{i,t}}\big\rangle \otimes \big|\psi_{E_0}\big\rangle\Big)\bigg)^\dagger} \tag{5.151}$$

where of course $\hat{\mathbf{U}}_t := e^{-it\gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}}$. It therefore follows that

$$\underline{\Big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\Big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}} \otimes \hat{\boldsymbol{\rho}}^{E_0}\big)\Big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\Big)} = \tag{5.152}$$

$$\underline{\bigg(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\hat{\mathbf{U}}_t\Big(\big|\psi_{S_{i,t}}\big\rangle \otimes \big|\psi_{E_0}\big\rangle\Big)\bigg)\bigg(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\hat{\mathbf{U}}_t\Big(\big|\psi_{S_{i,t}}\big\rangle \otimes \big|\psi_{E_0}\big\rangle\Big)\bigg)^\dagger} \tag{5.153}$$

We are now ready to present the main theorem of this section.

Theorem 5.4.1 (Estimating the Diagonal Terms (5.25))

$$\min_{PVM} \frac{1}{2}\bigg\| \sum_i \Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\Big) - \frac{1}{\mathscr{N}(t)}\sum_i \Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \hat{\mathbf{P}}_i^{E_t}\Big) \bigg\|_1 \leq \tag{5.154}$$

$$\min_{PVM} 4\sqrt{\sum_i \bar{p}_i(t)\bigg(1 - Tr\Big\{\hat{\mathbf{P}}_i^{E_t}\Lambda_{t,i}\Big(\big|\psi_{E_0}\big\rangle\big\langle\psi_{E_0}\big|\Big)\hat{\mathbf{P}}_i^{E_t}\Big\}\bigg)} \tag{5.155}$$

Here we have defined $\Lambda_{i,t}$ as follows.

$$\Lambda_{i,t}\big(\hat{\boldsymbol{\rho}}\big) := \int |\psi_{S_{i,t}}(x)|^2\Big(e^{-it\gamma x\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{it\gamma x\hat{\mathbf{B}}}\Big)dx \tag{5.156}$$

Proof. First, we will compute the following traces. Recall that $K_{i,t}(x,y) := \psi_{S_{i,t}}(x)\psi^*_{S_{i,t}}(y)$

$$\mathscr{N}_i(t) := Tr\bigg\{\Big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\Big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}} \otimes \hat{\boldsymbol{\rho}}^{E_0}\big)\Big(\mathbb{I} \otimes \hat{\mathbf{P}}_i^{E_t}\Big)\bigg\} = \tag{5.157}$$

$$\big\langle\psi_{S_{i,t}}\big|\otimes\big\langle\psi_{E_0}\big|\hat{\mathbf{U}}_t^\dagger\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\hat{\mathbf{U}}_t\big|\psi_{S_{i,t}}\big\rangle\otimes\big|\psi_{E_0}\big\rangle= \tag{5.158}$$

$$\left(\int\psi^*_{S_{i,t}}(y)\langle y|dy\otimes\big\langle\psi_{E_0}\big|\right)\hat{\mathbf{U}}_t^\dagger\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\hat{\mathbf{U}}_t\left(\int\psi_{S_{i,t}}(x)|x\rangle dx\otimes\big|\psi_{E_0}\big\rangle\right)= \tag{5.159}$$

$$\int\int\psi^*_{S_{i,t}}(x)\psi_{S_{i,t}}(y)\langle y|x\rangle\Big(\big\langle\psi_{E_0}\big|e^{it\gamma y\hat{\mathbf{B}}}\hat{\mathbf{P}}_i^{E_t}e^{-it\gamma x\hat{\mathbf{B}}}\big|\psi_{E_0}\big\rangle\Big)dxdy= \tag{5.160}$$

$$\int\big|\psi_{S_{i,t}}(x)\big|^2\Big(\big\langle\psi_{E_0}\big|e^{it\gamma y\hat{\mathbf{B}}}\hat{\mathbf{P}}_i^{E_t}e^{-it\gamma x\hat{\mathbf{B}}}\big|\psi_{E_0}\big\rangle\Big)dx= \tag{5.161}$$

$$\int\big|\psi_{S_{i,t}}(x)\big|^2\Big(\big\langle\psi_{E_0}\big|e^{it\gamma y\hat{\mathbf{B}}}\hat{\mathbf{P}}_i^{E_t}\hat{\mathbf{P}}_i^{E_t}e^{-it\gamma x\hat{\mathbf{B}}}\big|\psi_{E_0}\big\rangle\Big)dx= \tag{5.162}$$

$$\int|\psi_{S_{i,t}}(x)|^2\Big(Tr\{\hat{\mathbf{P}}_i^{E_t}e^{-it\gamma x\hat{\mathbf{B}}}\big|\psi_{E_0}\big\rangle\big\langle\psi_{E_0}\big|e^{it\gamma x\hat{\mathbf{B}}}\hat{\mathbf{P}}_i^{E_t}\}\Big)dx= \tag{5.163}$$

$$Tr\left\{\hat{\mathbf{P}}_i^{E_t}\left(\int|\psi_{S_{i,t}}(x)|^2\Big(e^{-it\gamma x\hat{\mathbf{B}}}\big|\psi_{E_0}\big\rangle\big\langle\psi_{E_0}\big|e^{it\gamma x\hat{\mathbf{B}}}\Big)dx\right)\hat{\mathbf{P}}_i^{E_t}\right\}= \tag{5.164}$$

$$Tr\left\{\hat{\mathbf{P}}_i^{E_t}\Lambda_{i,t}\Big(\big|\psi_{E_0}\big\rangle\big\langle\psi_{E_0}\big|\Big)\hat{\mathbf{P}}_i^{E_t}\right\} \tag{5.165}$$

Here the quantum map $\Lambda_{i,t}$ is defined as follows.

$$\Lambda_{i,t}\big(\hat{\boldsymbol{\rho}}\big):=\int|\psi_{S_{i,t}}(x)|^2\Big(e^{-it\gamma x\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}e^{it\gamma x\hat{\mathbf{B}}}\Big)dx \tag{5.166}$$

Now let us compute the following trace distance via an employment of Lemma 2.3.1.

$$\left\|\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)-\frac{1}{\mathscr{N}_i(t)}\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\right\|_1= \tag{5.167}$$

$$\left\|\Big(\hat{\mathbf{U}}_t\Big(\big|\psi_{S_{i,t}}\big\rangle\otimes\big|\psi_{E_0}\big\rangle\Big)\Big)\Big(\hat{\mathbf{U}}_t\Big(\big|\psi_{S_{i,t}}\big\rangle\otimes\big|\psi_{E_0}\big\rangle\Big)\Big)^\dagger-\left(\frac{\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\hat{\mathbf{U}}_t\Big(\big|\psi_{S_{i,t}}\big\rangle\otimes\big|\psi_{E_0}\big\rangle\Big)}{\sqrt{\mathscr{N}_i(t)}}\right)\left(\frac{\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\hat{\mathbf{U}}_t\Big(\big|\psi_{S_{i,t}}\big\rangle\otimes\big|\psi_{E_0}\big\rangle\Big)}{\sqrt{\mathscr{N}_i(t)}}\right)^\dagger\right\|_1= \tag{5.168}$$

$$2\sqrt{1-\left|\frac{\big\langle\psi_{S_{i,t}}\big|\otimes\big\langle\psi_{E_0}\big|\hat{\mathbf{U}}_t^\dagger\Big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\Big)\hat{\mathbf{U}}_t\big|\psi_{S_{i,t}}\big\rangle\otimes\big|\psi_{E_0}\big\rangle}{\sqrt{\mathscr{N}_i(t)}}\right|^2}= \tag{5.169}$$

$$2\sqrt{1-\left|\frac{\mathscr{N}_i(t)}{\sqrt{\mathscr{N}_i(t)}}\right|^2}=2\sqrt{1-\mathscr{N}_i(t)} \tag{5.170}$$

Recapitulating, we have

$$\mathscr{N}_i(t)=Tr\left\{\hat{\mathbf{P}}_i^{E_t}\Lambda_{i,t}\Big(\big|\psi_{E_0}\big\rangle\big\langle\psi_{E_0}\big|\Big)\hat{\mathbf{P}}_i^{E_t}\right\} \tag{5.171}$$

and

$$\left\|\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)-\frac{1}{\mathscr{N}_i(t)}\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\right\|_1=2\sqrt{1-\mathscr{N}_i(t)} \tag{5.172}$$

Using Corollary (5.4.1), our only task in proving Theorem 5.4.1 will be to estimate (5.144). Using

(5.171) and (5.172) we get the following.

$$\frac{1}{2}\sum_i \bar{p}_i(t)\left\|\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)-\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\right\|_1\leq \tag{5.173}$$

$$\frac{1}{2}\sum_i \bar{p}_i(t)\left[\left\|\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)-\frac{1}{\mathscr{N}_i(t)}\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\right\|_1+\right. \tag{5.174}$$

$$\left.\frac{1}{2}\left\|\frac{1}{\mathscr{N}_i(t)}\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)-\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\right\|_1\right]= \tag{5.175}$$

$$\frac{1}{2}\sum_i \bar{p}_i(t)\left[\left\|\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)-\frac{1}{\mathscr{N}_i(t)}\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\right\|_1+\left|\frac{1}{\mathscr{N}_i(t)}-1\right|\left\|\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\right\|_1\right]= \tag{5.176}$$

$$\frac{1}{2}\sum_i \bar{p}_i(t)\left[\left\|\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)-\frac{1}{\mathscr{N}_i(t)}\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\mathscr{U}_t\big(\hat{\boldsymbol{\rho}}_{S_{i,t}}\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\big(\mathbb{I}\otimes\hat{\mathbf{P}}_i^{E_t}\big)\right\|_1+\left|\frac{1}{\mathscr{N}_i(t)}-1\right|\mathscr{N}_i(t)\right]= \tag{5.177}$$

$$\sum_i \bar{p}_i(t)\left[2\sqrt{1-\mathscr{N}_i(t)}+1-\mathscr{N}_i(t)\right]\leq\sum_i \bar{p}_i(t)\left[2\sqrt{1-\mathscr{N}_i(t)}\right]= \tag{5.178}$$

$$2\sum_i \bar{p}_i(t)\sqrt{1-\mathscr{N}_i(t)}\leq 2\sqrt{\sum_i \bar{p}_i(t)\big(1-\mathscr{N}_i(t)\big)}= \tag{5.179}$$

$$2\sqrt{\sum_i \bar{p}_i(t)\left(1-Tr\left\{\hat{\mathbf{P}}_i^{E_t}\Lambda_{i,t}\Big(\big|\psi_{E_0}\big\rangle\big\langle\psi_{E_0}\big|\Big)\hat{\mathbf{P}}_i^{E_t}\right\}\right)} \tag{5.180}$$

Here we have employed Jensen's inequality for concave functions in (5.179).

By virtue of Corollary 5.4.1 we therefore have

$$\frac{1}{2}\left\|\sum_i\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\mathbb{I}\Big)-\sum_i\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\hat{\mathbf{P}}_i^{E_t}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\hat{\mathbf{P}}_i^{E_t}\Big)\right\|_1\leq \tag{5.181}$$

$$4\sqrt{\sum_i \bar{p}_i(t)\left(1-Tr\left\{\hat{\mathbf{P}}_i^{E_t}\Lambda_{i,t}\Big(\big|\psi_{E_0}\big\rangle\big\langle\psi_{E_0}\big|\Big)\hat{\mathbf{P}}_i^{E_t}\right\}\right)} \tag{5.182}$$

Finally, a simple application of Lemma 4.4.1 leads to

$$\frac{1}{2}\left\|\sum_i\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\mathbb{I}\Big)-\frac{1}{\mathscr{N}(t)}\sum_i\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\hat{\mathbf{P}}_i^{E_t}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\hat{\mathbf{P}}_i^{E_t}\Big)\right\|_1\leq \tag{5.183}$$

$$4\sqrt{\sum_i \bar{p}_i(t)\left(1-Tr\left\{\hat{\mathbf{P}}_i^{E_t}\Lambda_{i,t}\Big(\big|\psi_{E_0}\big\rangle\big\langle\psi_{E_0}\big|\Big)\hat{\mathbf{P}}_i^{E_t}\right\}\right)} \tag{5.184}$$

Taking the minimum over all PVM acting on the environmental degrees of freedom on both sides of inequality (5.183) (5.184) we get the result we set out to prove. □

Theorem 5.4.1 may be easily generalized to support the setting where N_E environments are present and M_E environments have been traced out as previously mentioned in the discussion spanning equations (5.3) through (5.11). We present this without proof since the steps are analogous to all of the steps involved in proving Theorem 5.4.1.

Theorem 5.4.2 (Estimating the Diagonal Terms (5.25) for N_E Environments)

$$\min_{PVM}\frac{1}{2}\left\|\sum_i\left(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\mathbb{I}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\mathbb{I}\right)-\frac{1}{\mathcal{N}(t)}\sum_i\left(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right)\right\|_1\leq \tag{5.185}$$

$$\min_{PVM}2\sqrt{\sum_i\bar{p}_i(t)\left(1-Tr\left\{\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\Lambda_{i,t}\left(\bigotimes_{k=1}^{N_E}\left|\psi_{E_0^k}\right\rangle\left\langle\psi_{E_0^k}\right|\right)\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right\}\right)} \tag{5.186}$$

Here we have defined $\Lambda_{i,t}$ as follows.

$$\Lambda_{i,t}\left(\hat{\boldsymbol{\rho}}\right):=\int|\psi_{S_{i,t}}(x)|^2\left(e^{-itx\sum_{k=1}^{N_E}g_k\hat{\mathbf{B}}_k}\hat{\boldsymbol{\rho}}e^{itx\sum_{k=1}^{N_E}g_k\hat{\mathbf{B}}_k}\right)dx \tag{5.187}$$

All of the $\hat{\mathbf{B}}_k$ act on their respective Hilbert space.

A little thought convinces one that if we were to constrain ourselves to the case where all of the $\left|\psi_{E_0^k}\right\rangle\left\langle\psi_{E_0^k}\right|$ are identical and all of the $g_k\hat{\mathbf{B}}_k$ are identical, then Theorem (5.4.1) would take the following simpler form.

Corollary 5.4.2 (Estimating the Diagonal Terms (5.25) for N_E identical Environments with identical $\hat{\mathbf{B}}_k$ and identical g_k)

$$\min_{PVM}\frac{1}{2}\left\|\sum_i\left(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\mathbb{I}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\mathbb{I}\right)-\frac{1}{\mathcal{N}(t)}\sum_i\left(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right)\hat{\boldsymbol{\rho}}_t\left(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\bigotimes_{k=1}^{N_E}\hat{\mathbf{P}}_i^{E_t^k}\right)\right\|_1\leq \tag{5.188}$$

$$\min_{PVM}2\sqrt{\sum_i\bar{p}_i(t)\left(1-\int|\psi_{S_{i,t}}(x)|^2\left\langle\psi_{E_t}(x)\right|\hat{\mathbf{P}}_i^{E_t}\left|\psi_{E_t}(x)\right\rangle^{N_E}dx\right)} \tag{5.189}$$

where

$$\left|\psi_{E_t}(x)\right\rangle:=e^{-itxg\hat{\mathbf{B}}}\left|\psi_{E_0}\right\rangle \tag{5.190}$$

Notice that the object in the square root of (5.189) is the general case of the bound encountered in (3.188) where $N_E=1$, the ambient Hilbert space was $L^2(\mathbb{R})$ and the equivalent in (3.188) of the kernel $K(x,x)$ was compactly supported; this in turn lead to a finite set of partitions $\Delta_{i,t}$ and a specific PVM $\left\{\hat{\mathbf{P}}_i^{E_t}\right\}_i$ which grew more optimal (in the sense of fully solving the respective QSD problem) as t grew large.

Theorems 5.4.1 and 5.4.2, and Corollary 5.4.2 are tools that may aid in the estimation of the diagonal term (5.25). The drawback to these bounds is that they require one to find an approximately optimal PVM acting on the environmental degrees of freedom, this is in contrast with the discrete variables, where we devised a PVM independent bound (4.2.2). It is important to note that the density operators $\Lambda_{t,i}\left(\hat{\boldsymbol{\rho}}^{E_0}\right)$ are not pure, we may therefore not apply Theorem 4.2.2 in this case. In order for 4.2.2 to be applicable, the $\Lambda_{i,t}\left(\hat{\boldsymbol{\rho}}^{E_0}\right)$ must be approximately pure. To see this let us consider the term in the square root of (5.186). Now, define $\hat{\boldsymbol{\rho}}_{x_i}^{E_t}:=e^{-it\gamma x_i\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}^{E_0}e^{it\gamma x_i\hat{\mathbf{B}}}$, where

$x_i := \int x|\psi_{S_{i,t}}(x)|^2 dx$. We have the following.

$$1 - Tr\{\hat{\mathbf{P}}_i^{E_t}\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0})\hat{\mathbf{P}}_i^{E_t}\} \leq \tag{5.191}$$

$$\left\|\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0}) - \hat{\mathbf{P}}_i^{E_t}\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0})\hat{\mathbf{P}}_i^{E_t}\right\|_1 = \tag{5.192}$$

$$\left\|\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0}) - \hat{\boldsymbol{\rho}}_{x_i}^{E_t} + \hat{\boldsymbol{\rho}}_{x_i}^{E_t} - \hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t} + \hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t} - \hat{\mathbf{P}}_i^{E_t}\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0})\hat{\mathbf{P}}_i^{E_t}\right\|_1 \leq \tag{5.193}$$

$$\left\|\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0}) - \hat{\boldsymbol{\rho}}_{x_i}^{E_t}\right\|_1 + \left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t} - \hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t}\right\|_1 + \left\|\hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t} - \hat{\mathbf{P}}_i^{E_t}\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0})\hat{\mathbf{P}}_i^{E_t}\right\|_1 \leq \tag{5.194}$$

$$\left\|\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0}) - \hat{\boldsymbol{\rho}}_{x_i}^{E_t}\right\|_1 + \left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t} - \hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t}\right\|_1 + \left\|\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0}) - \hat{\boldsymbol{\rho}}_{x_i}^{E_t}\right\|_1 = \tag{5.195}$$

$$2\left\|\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0}) - \hat{\boldsymbol{\rho}}_{x_i}^{E_t}\right\|_1 + \left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t} - \hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t}\right\|_1 = \tag{5.196}$$

With this result, we now bound (5.186).

$$2\min_{PVM}\sqrt{\sum_i \bar{p}_i(t)\Big(1 - Tr\{\hat{\mathbf{P}}_i^{E_t}\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0})\hat{\mathbf{P}}_i^{E_t}\}\Big)} \leq \tag{5.197}$$

$$2\min_{PVM}\sqrt{\sum_i \bar{p}_i(t)\Big(2\left\|\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0}) - \hat{\boldsymbol{\rho}}_{x_i}^{E_t}\right\|_1 + \left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t} - \hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t}\right\|_1\Big)} \leq \tag{5.198}$$

$$2\sqrt{2\sum_i \bar{p}_i(t)\left\|\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0}) - \hat{\boldsymbol{\rho}}_{x_i}^{E_t}\right\|_1} + 2\min_{PVM}\sqrt{\sum_i \bar{p}_i(t)\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t} - \hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t}\right\|_1} \tag{5.199}$$

We will write the latter result as a lemma.

Lemma 5.4.3 (Diagonal terms of the SBS problem for continuous variables, further estimates)

$$\min_{PVM}\frac{1}{2}\left\|\sum_i\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\mathbb{I}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\mathbb{I}\Big) - \frac{1}{\mathcal{N}(t)}\sum_i\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\hat{\mathbf{P}}_i^{E_t}\Big)\hat{\boldsymbol{\rho}}_t\Big(\hat{\mathbf{P}}_{\Delta_{i,t}}\otimes\hat{\mathbf{P}}_i^{E_t}\Big)\right\|_1 \leq \tag{5.200}$$

$$4\sqrt{2\sum_i \bar{p}_i(t)\left\|\Lambda_{i,t}(\hat{\boldsymbol{\rho}}^{E_0}) - \hat{\boldsymbol{\rho}}_{x_i}^{E_t}\right\|_1} + 4\min_{PVM}\sqrt{\sum_i \bar{p}_i(t)\left\|\hat{\boldsymbol{\rho}}_{x_i}^{E_t} - \hat{\mathbf{P}}_i^{E_t}\hat{\boldsymbol{\rho}}_{x_i}^{E_t}\hat{\mathbf{P}}_i^{E_t}\right\|_1} \tag{5.201}$$

This can be easily extended to the case where we have more than one environmental degree of freedom. In such a case, Lemma 5.4.3 becomes the following.

Corollary 5.4.3 (Diagonal terms for continuous variables N_E macro-environment case; further estimates)

$$\min_{PVM} \frac{1}{2}\left\| \sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right) - \frac{1}{\mathcal{N}(t)} \sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k}\right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k}\right) \right\|_1 \leq \tag{5.202}$$

$$4\sqrt{2\sum_i \bar{p}_i(t) \left\| \Lambda_{i,t}\left(\bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_0^k}\right) - \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \right\|_1} + 4 \min_{PVM} \sqrt{\sum_i \bar{p}_i(t) \left\| \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1} \tag{5.203}$$

Using Corollary 5.4.3 and Lemma 4.1, we obtain the following useful corollary.

Corollary 5.4.4 (Further estimates)

$$\frac{1}{2} \min_{PVM} \left\| \sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \mathbb{I}\right) - \frac{1}{\mathcal{N}(t)} \sum_i \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k}\right) \hat{\boldsymbol{\rho}}_t \left(\hat{\mathbf{P}}_{\Delta_{i,t}} \otimes \bigotimes_{k=1}^{N_E} \hat{\mathbf{P}}_i^{E_t^k}\right) \right\|_1 \leq \tag{5.204}$$

$$4\sqrt{2\sum_i \bar{p}_i(t) \sum_{k=1}^{N_E} \int |\psi_{S_{i,t}}(x)|^2 \left\| \hat{\boldsymbol{\rho}}_x^{E_t^k} - \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \right\|_1 dx} + 4 \min_{PVM} \sqrt{\sum_i \bar{p}_i(t) \sum_{k=1}^{N_E} \left\| \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} - \hat{\mathbf{P}}_i^{E_t^k} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \hat{\mathbf{P}}_i^{E_t^k} \right\|_1} \tag{5.205}$$

Proof. First note that

$$\left\| \Lambda_{i,t}\left(\bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}^{E_0^k}\right) - \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \right\|_1 = \left\| \int |\psi_{S_{i,t}}(x)|^2 \left(\bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_x^{E_t^k}\right) dx - \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k} \right\|_1 = \tag{5.206}$$

$$\left\| \int |\psi_{S_{i,t}}(x)|^2 \left(\bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_x^{E_t^k} - \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\right) dx \right\|_1 \leq \tag{5.207}$$

$$\int |\psi_{S_{i,t}}(x)|^2 \left\| \left(\bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_x^{E_t^k} - \bigotimes_{k=1}^{N_E} \hat{\boldsymbol{\rho}}_{x_i}^{E_t^k}\right) \right\|_1 dx. \tag{5.208}$$

Using the latter, the proof follows directly from Lemma 4.1 and Theorem 5.4.3 by noting that

$$\left\| \hat{\boldsymbol{\rho}}_x^{E_t^k} \right\|_1 = 1 \tag{5.209}$$

for all t, k and x. □

If the first term of (5.205) is small then we may benefit from the use of Theorem 4.2.2 in estimating the second term of (5.205).

5.4.1 How General Can $\hat{\mathbf{X}}$ Be?

The operator $\hat{\mathbf{X}}$ may be taken to be a more general self-adjoint operator; so long as it has purely continuous spectrum, and in particular a non-empty Rajchman subspace, then all of the above work of this chapter has an analog with some modifications. To see this we write $\hat{\mathbf{X}}$ in its spectral decomposition form, using the spectral theorem, i.e.

$$\hat{\mathbf{X}} = \int_{\sigma(\hat{\mathbf{X}})} \lambda d\hat{\mathbf{E}}_\lambda \tag{5.210}$$

where $d\hat{\mathbf{E}}_\lambda$ is a PVM. Now, notice that the operator (5.30) may be expressed as

$$\big(e^{-it\gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}}\big)\big(\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\big)\otimes\hat{\boldsymbol{\rho}}^{E_0}\big)\big(e^{it\gamma\hat{\mathbf{X}}\otimes\hat{\mathbf{B}}}\big) = \tag{5.211}$$

$$\int\int \frac{d\hat{\mathbf{E}}_\lambda}{d\lambda}\mathscr{E}_t\big(\hat{\boldsymbol{\rho}}_{S_0}\big)\frac{d\hat{\mathbf{E}}_{\lambda'}}{d\lambda'}\otimes e^{-it\gamma\lambda\hat{\mathbf{B}}}\hat{\boldsymbol{\rho}}^{E_0}e^{it\gamma\lambda'\hat{\mathbf{B}}}d\lambda d\lambda'. \tag{5.212}$$

Using this operator and developing the respective appropriate partition, as we did for the position operator in (5.36), we may attain analogs to all of the theorems in the previous subsections of this chapter. This includes Theorem 5.4.2. Care must be taken in making sense of the terms $\frac{d\hat{\mathbf{E}}_\lambda}{d\lambda}$ however. For the case where $\hat{\mathbf{X}}$ is simply assumed to be a multiplication operator with a purely absolutely continuous spectrum, the treatment becomes virtually identical to that of the case where $\hat{\mathbf{X}}$ is a position operator which was the focus of this chapter. In future work, we might push toward further generalities.

Concluding Remarks and Future Work

In this work the central object of study has been

$$Tr_{E_{N_E+1},E_{N_E+2},\dots,E_N}\left\{e^{-it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N}g_k\hat{\mathbf{B}}_k}\left(\hat{\boldsymbol{\rho}}_{S_0}\otimes\bigotimes_{k=1}^{N}\hat{\boldsymbol{\rho}}^{E_0^k}\right)e^{it\hat{\mathbf{X}}\otimes\sum_{k=1}^{N}g_k\hat{\mathbf{B}}_k}\right\}. \tag{5.213}$$

In Chapter 5 we have made significant strides in defining what the associated SBS state of (5.213) should be (SBSCV), and developing techniques for estimating the proximity between (5.213) and its associated SBS state. We have done the analogous work in Chapter 4 for the case of discrete variables; albeit a formulation of SBS for discrete variables predates this dissertation. However, there is much left to be desired as the form of (5.213) is rather restrictive. In future work, we would like to consider the cases where the quantum-measurement limit and the von Neumann interaction operator assumptions are relaxed. i.e. we would like to consider the case where $\hat{\mathbf{H}}_{tot}$ is not approximately $\hat{\mathbf{H}}_{int}$ and the interaction term $\hat{\mathbf{H}}_{int}$ does not have the tensor product form (von Neumann interaction) $\hat{\mathbf{X}}\otimes\sum_{k=1}^{N}g_k\hat{\mathbf{B}}_k$. It would be interesting to find out for which families of $\hat{\mathbf{H}}_{tot}$ one is able to prove that there exist time domains for which the dynamics push the state

$$\hat{\boldsymbol{\rho}}_t := Tr_{E_{N_E+1},E_{N_E+2},\dots,E_N}\left\{e^{-it\hat{\mathbf{H}}_{tot}}\left(\hat{\boldsymbol{\rho}}_{S_0}\otimes\bigotimes_{k=1}^{N}\hat{\boldsymbol{\rho}}^{E_0^k}\right)e^{it\hat{\mathbf{H}}_{tot}}\right\}. \tag{5.214}$$

into an SBS/SBSCV regime. Of course in this case estimating the respective SBS proximity will be much more daunting since we will in general not be able to execute the decompositions (5.4) (4.40) which were key in estimating the SBS and SBSCV problems for the von Neuman type interactions case in Chapters 4 and 5. If progress could be made in estimating 5.214 with a more general $\hat{\mathbf{H}}_{tot}$, then a next step of interest would be to relax the separable initial state condition. i.e. rather than considering the case where $\hat{\boldsymbol{\rho}}_{S_0}\otimes\bigotimes_{k=1}^{N}\hat{\boldsymbol{\rho}}^{E_0^k}$ as our total initial state, we could consider an arbitrary $\hat{\boldsymbol{\rho}}_0\in\mathcal{S}\left(\mathscr{H}_S\otimes\bigotimes_{k=1}^{N_E}\mathscr{H}_{E^k}\right)$ as the total initial state. It would be very interesting to prove that SBS states arise for a broader family of dynamics and initial states.

As a parting remark, we heuristically discuss what we have learned from our SBS studies for (5.214). We have learned that even simple purely decoherent dynamics (no dissipation/ no exchange of kinetic energy) is enough to render classical objectivity (Definition 4.1.1) from the quantum. Macroscopic systems such as human beings are perpetually scattering smaller particles such as air molecules, and photons; scattering events that may be modeled by recoilless-scattering-models such as those afforded by von Neumann-type Hamiltonians (1.98). It would seem that classicality at the macroscale arises from the quantum due to the gargantuan quantity of interactions classical objects have with their environments. It would therefore seem that as classical beings we can never find ourselves in a superposition whilst living. These are indeed questions for physicists to grapple with and beyond the scope of this dissertation but thought-provoking and not fully out of place. Rather than concluding this dissertation with a thousand words we shall conclude it with a picture. We present in the following page an artistic interpretation of quantum to classical transitions as characterized by SBS theory (Figure 5.1).

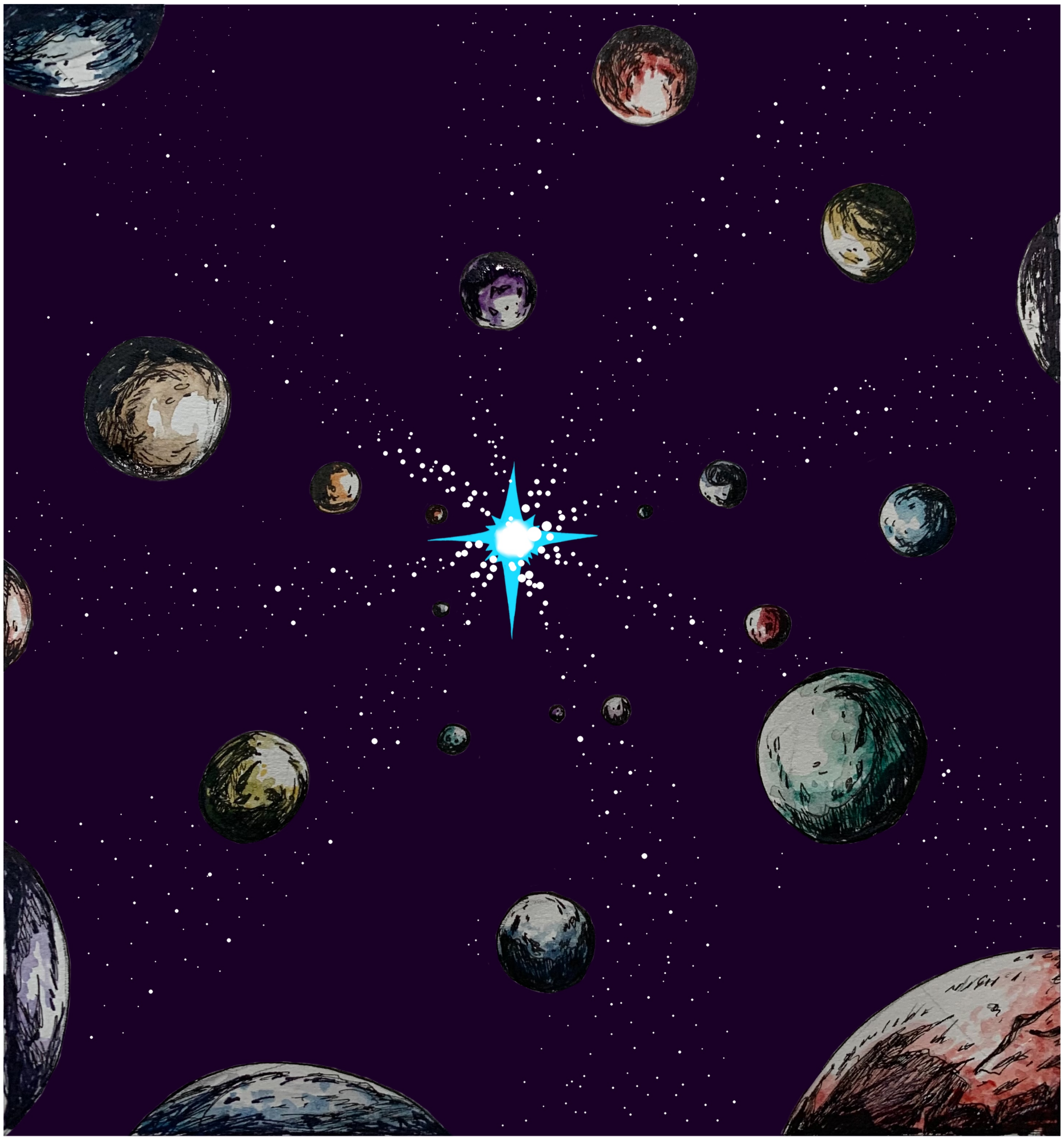

Figure 5.1: *An artistic interpretation of quantum to classical transitions as characterized by SBS theory. The author of this thesis described SBS theory to artist Timothy Martinez (@timbosculpt) and this beautiful artistic interpretation resulted. The author omits his own interpretation of this gorgeous work, leaving the reader to insert their own.*

Appendix A: Notation

Herein we present some of the notational conventions of the Thesis in order to aid the reader.

- $\hat{\mathbf{A}}$:= operator.
- $\hat{\boldsymbol{\rho}}$:= Density operator (Definition 1.3.2).
- $\hat{\boldsymbol{\rho}}_{dg,t}$:= See (4.50)
- $\hat{\boldsymbol{\rho}}_{SBS,t}$:= See (4.51)
- $\hat{\boldsymbol{\rho}}_{PSBS,t} := \mathscr{N}\hat{\boldsymbol{\rho}}_{SBS,t}$
- $Tr\{\hat{\mathbf{A}}\}$:= Trace of some trace class operator $\hat{\mathbf{A}}$. See Definition 1.3.3.
- $Tr_{E^k}\{\hat{\mathbf{A}}\}$:= Partial trace of the kth environmental degrees of freedom. See Definition 1.4.1.
- $\mathcal{S}(\mathscr{H})$:= Space of density operators acting in $\mathscr{H}$.
- $\mathscr{H}$:= Hilbert Space.
- $\mathscr{H}_S$:= Hilbert Space of the system.
- $\mathscr{H}_{E^k}$:= Hilbert Space of kth Environment.
- $\hat{\mathbf{H}}$:= Hamiltonian.
- $\hat{\mathbf{H}}_{int}$:= Interaction Hamiltonian.
- $|\psi\rangle$:= Vector in some Hilbert Space.
- $\hat{\mathbf{U}}_t$:= Unitary operator.
- $\hat{\mathbf{H}}_0 := -\sum_k \frac{1}{2m_k}\partial^2_{x_k}$
- $\mathbf{Spec}\{\hat{\mathbf{A}}\}$:= Spectrum of an operator $\hat{\mathbf{A}}$. See Definition 1.3.1.
- $\mathscr{E}(\hat{\boldsymbol{\rho}})$: Quantum map. See Definition 2.1.1.
- $\hat{\mathbf{M}}$:= Krauss operators 1.75.
- $\|\hat{\mathbf{A}}\|_1 := Tr\left\{\sqrt{\hat{\mathbf{A}}^\dagger\hat{\mathbf{A}}}\right\}$
- $F(\hat{\boldsymbol{\rho}},\hat{\boldsymbol{\sigma}}) := \left\|\sqrt{\hat{\boldsymbol{\rho}}}\sqrt{\hat{\boldsymbol{\sigma}}}\right\|_1^2$
- POVM and PVM:= Projective operator valued measure and projector valued measure respectively. See 2.2.3 a discussion.
- p_E:= Shorthand for

  $$\min_{POVM} p_E\left\{\{p_i,\hat{\boldsymbol{\rho}}_i\}_{i=1}^N,\{\hat{\mathbf{M}}_l\}_{l=1}^N\right\} :=$$

  $\min_{POVM}\left\{1-\sum_{i=1}^N p_i Tr\left\{\hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}_i\hat{\boldsymbol{M}}_i^\dagger\right\}\right\}$.
- QSD:= Quantum state discrimination. See the intro to Chapter 3.
- $D^k_{s,t}$:= See (4.69).

2

Appendix B: Gap in Corollary 1 of [40]

Let $\left\{\hat{\mathbf{E}}_l\right\}_l$ be a POVM. The operators $\hat{\mathbf{E}}_l$ act over some unspecified Hilbert space. Indeed, $\sum_l \hat{\mathbf{E}}_l = \mathbb{I}$, $\|\hat{\mathbf{E}}_l\| \leq 1$ and $\hat{\mathbf{E}}_l$ are positive operators that may be written as $\hat{\mathbf{E}}_l = \hat{\mathbf{M}}_l^\dagger \hat{\mathbf{M}}_l$ where $\hat{\mathbf{M}}_l$ are bounded operators. Now, let $\hat{\boldsymbol{\rho}}$ be a density operator acting over the same Hilbert space as the POVM $\left\{\hat{\mathbf{E}}_l\right\}_l$. A question arises regarding the positive semidefiniteness of the operator $\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_l \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_l^\dagger$.

Claim 5.4.1 (Non positivity of a particular operator)

$\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_l \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_l^\dagger$ is not positive semidefinite in general

Proof. Counter example.

Consider the 2 dimensional case where $\hat{\boldsymbol{\rho}} = \begin{pmatrix} 1-\delta & 0 \\ 0 & \delta \end{pmatrix}$ $(0 \leq \delta \leq 1)$ and we have a POVM characterized by the operator $\hat{\mathbf{M}}_0 = a \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$ $(a < 1)$ which is a scaled projector. The PVOM in question is

$$\left\{\hat{\mathbf{M}}_0^\dagger \hat{\mathbf{M}}_0,\ \mathbb{I} - \hat{\mathbf{M}}_0^\dagger \hat{\mathbf{M}}_0\right\}. \tag{5.215}$$

Let us take a look at the operator

$$\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0 \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_0 \tag{5.216}$$

Expanding things out this looks as follows; in matrix notation.

$$\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0 \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_0 = \begin{pmatrix} 1-\delta & 0 \\ 0 & \delta \end{pmatrix} - \frac{a^2}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1-\delta-\frac{a^2}{4} & -\frac{a^2}{4} \\ -\frac{a^2}{4} & \delta - \frac{a^2}{4} \end{pmatrix}. \tag{5.217}$$

For this operator to be positive semidefinite we require that $\langle\phi|\left\{\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0 \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_0\right\}|\phi\rangle \geq 0$ hold for all $|\phi\rangle$ in the Hilbert space in question. Let us use the unit vector $\tilde{\mathbf{e}}_2 = (0,1)^t$. In this case

$$\langle\tilde{\mathbf{e}}|\left\{\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0 \hat{\boldsymbol{\rho}} \hat{\mathbf{M}}_0\right\}|\tilde{\mathbf{e}}\rangle = \delta - \frac{a^2}{4} \tag{5.218}$$

But notice that if $\delta < \frac{a^2}{4}$, which is a viable possibility, then we do not have positive definiteness for $\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_0$. □

That is it for the counter-example. Notice that in the case where $a = 1$, $\hat{\mathbf{M}}_0$ is a projector, and even then we do not have positive definiteness for $\hat{\boldsymbol{\rho}} - \hat{\mathbf{M}}_0\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_0$ in general since this breaks down for $\delta < \frac{1}{4}$.

Now, in the paper [40] the authors provide a proof for equation (5) on page 2 of said paper. This proof involves the computation of a trace distance of the form $\left\|\hat{\boldsymbol{\rho}} - \hat{\mathbf{P}}\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}\right\|_1$ (where $\hat{\mathbf{P}}$ is a projector) ; see page (1) of the appendix of the same paper and look at the sentence preceding equation (4) of page one of this appendix. There the authors implicitly argue that $\left\|\hat{\boldsymbol{\rho}} - \hat{\mathbf{P}}\hat{\boldsymbol{\rho}}\hat{\mathbf{P}}\right\|_1 = Tr\left\{\hat{\boldsymbol{\rho}}(\mathbb{I} - \hat{\mathbf{P}})\right\}$ in general. This, however, is only true if $\hat{\boldsymbol{\rho}} - \hat{\mathbf{P}}\hat{\boldsymbol{\rho}}\hat{\mathbf{P}} \geq 0$, and this in turn is true only when $\hat{\mathbf{P}}$ commutes with $\hat{\boldsymbol{\rho}}$. It looks like, tacitly, they are assuming that the PVMS, amongst other assumptions, have the special property that (now I use their notation) the $\hat{\mathbf{P}}_i$ projector commute with the $\hat{\boldsymbol{\rho}}_i$ terms of the mixture $\sum_i p_i\hat{\boldsymbol{\rho}}_i$ where $\hat{\mathbf{P}}_i$ is an element of a POVM used to discriminate the mixture $\sum_i p_i\hat{\boldsymbol{\rho}}$. This assumption however need not in general be true and the bound by Knill and Barnum [27] does not assume commutativity for their result that bounds the trace

$$Tr\left\{\sum_i p_i\hat{\boldsymbol{\rho}} - \sum_i \hat{\mathbf{M}}_i\hat{\boldsymbol{\rho}}\hat{\mathbf{M}}_i^\dagger\right\} \tag{5.219}$$

to hold when minimizing over appropriate POVM, $\left\{\hat{\mathbf{M}}_i\right\}_i$, schemes and neither do they assume that we discriminate with projectors, their result uses the objective function which minimizes over all POVM. This means that the assumption that $\hat{\mathbf{P}}_i$ commutes with $\hat{\boldsymbol{\rho}}_i$ makes the minimization calculated in [40] an upper bound to the one proven by Knill and Barnum [27]. Unfortunately starting from $\|\hat{\boldsymbol{\rho}}_i - \hat{\mathbf{P}}_i\hat{\boldsymbol{\rho}}_i\hat{\mathbf{P}}_i\|_1$ and bounding such an object by fidelities is significantly harder.

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